Chapter 13 - Inverse Functions

In the second part of this book on Calculus, we shall be devoting our study to another type of function, the exponential function and its close relative the Sine function. Before we immerse ourselves in this complex and analytical study, we first need to understand something about inverse functions.

The Inverse function is by definition a function who output becomes the input or the independent variable becomes an independent variable. For example given the function:

$$F(a) = ma$$

Which is Newton.s second law, or the force acting on a body of mass, m, is a function of the acceleration given to it. We are free to input any a and what we get out is the force. The inverse of this Force function, according to the definition, will give us the acceleration as a function of Force. This is done by simply solving for the independent variable, a:

$$F = ma$$

$$\frac{F}{m} = a$$

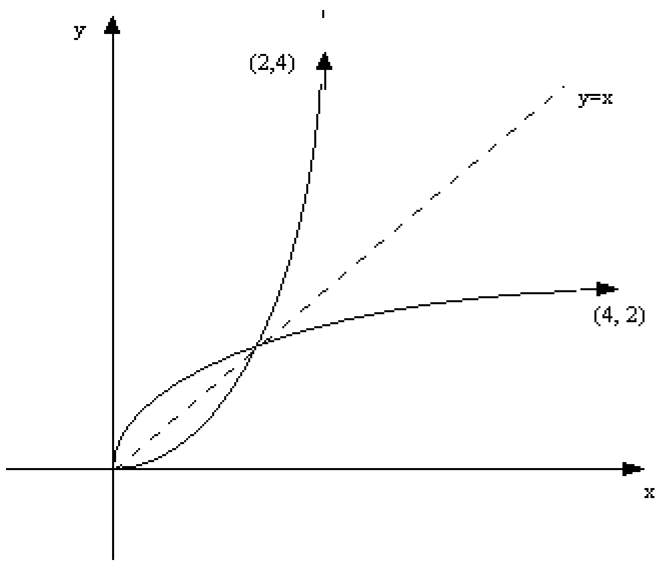
$$a(F) = \frac{F}{m}$$

Now I can let F be anything and then find the acceleration as a function of it.

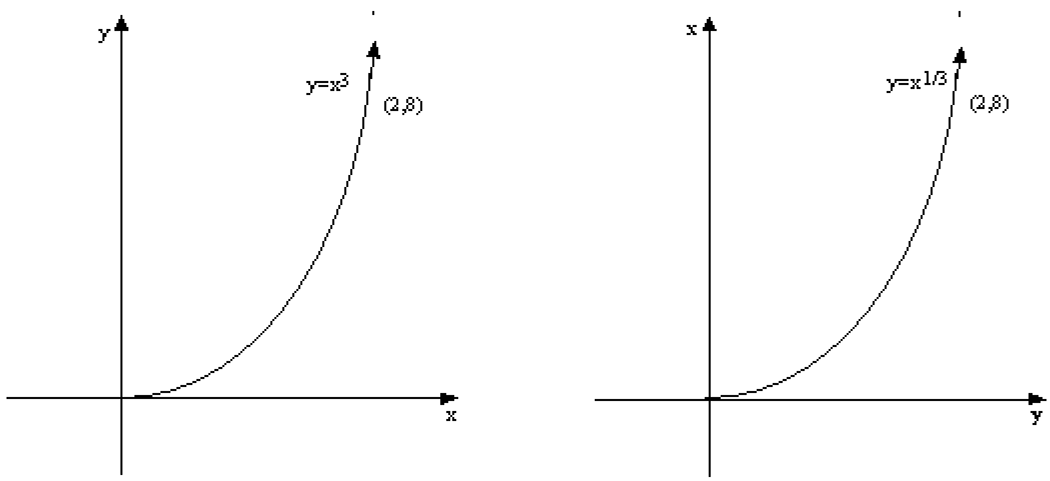
The inverse of a function, f(x), is commonly written as, $f^{-1}(x)$. Now we will look at the more general case of graphing a function and its inverse in the same co-ordinate planes. Given the function $y = x^3$, to calculate its inverse we only have to solve this for x to get $x = \sqrt[3]{y}^{\,1}$. Notice that we have not really changed the function at all, we have only solved for the independent variable. The graph of these two functions would be exactly the same. Our definition of the inverse function therefore has to be slightly modified. After finding the inverse of a function we just interchange x and y to get: $y = \sqrt[3]{x}^{\,1}$

What does this do to the inverse function? This essential flips the graph of f(x) about the line y=x such that for every point (x,y) there is a corresponding point (y,x) on the graph of the inverse function. Now both the functions can be graphed in the same x-y plane.

Remember that if we just solve for the dependent, we are not changing the equation but merely re-writing it. For this reason its graph is the same. By flipping the x and y, we get another function of x, whose relation to f (x) is that it has been graphed as though the x-axis were the y-axis and vice-versa. It is best we look at the two graphs:



Notice how every point (x, y) has a corresponding point (y,x) on the inverse function. The graph of the inverse function is therefore exactly the same as the original function except that the x and y-axis have been switched:



Since every point (x, y) has a corresponding point (y,x) then any point y from the inverse function when inputted in the original function should yield x:

$$f(x) = x^3 \text{ and } f^{-1}(x) = \sqrt[3]{x}$$

$$f(f^{-1}(x)) = x$$

$$f(\sqrt[3]{x}) = \left(\sqrt[3]{x}\right)^3 = x$$

Remember the a function and its inverse are both function.s of x. The way they are related is that the inverse function represents the original function by just having its dependent and independent variable switched around. As you can see from the first graph, when the two function.s are graphed together, the inverse function contains all the point (x, y), of the first function, plotted as (y, x) with the exception that y is given as function of x. For this reason

$$f^{-1}(f(x)) = x$$
$$f^{-1}(x^3) = \sqrt[3]{x^3} = x$$

What is important to understand about the inverse function is that it is obtained by solving for the independent variable, then replacing it with y, to create a function that is also a function of x and can be graphed along with the original function.

Now that we know how a function and its inverse function are closely related, it brings us to the question, how are the derivatives related? Logic would tell us that instead of $\frac{\Delta y}{\Delta x}$ we should just find $\frac{\Delta x}{\Delta y}$ by taking the reciprocal of the derivative. For example if we had:

$$y = x^2 \text{ or } f(x) = x^2$$
$$x = \sqrt{y}$$
$$y = \sqrt{x} \text{ or } f^{-1}(x) = \sqrt{x}$$

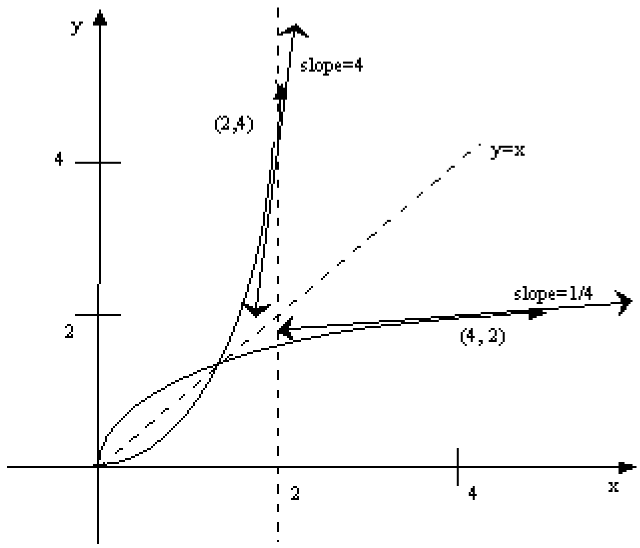
The derivative of the inverse function might be:

$$\frac{\Delta y}{\Delta x} = 2x$$
$$\frac{\Delta x}{\Delta y} = \frac{1}{\frac{\Delta y}{\Delta x}} = \frac{1}{2x}$$

Or the derivative of $f^{-1}(x) \text{ or } y = \sqrt{x}$ is 1/2x. But this is not the case, the derivative is:

$$y = \sqrt{x}$$
$$\frac{\Delta y}{\Delta x} = \frac{1}{2\sqrt{x}}$$

Let us examine the graph of f(x) and its inverse function to see what exactly is going on.



Note that at x=2, the slopes are not reciprocals but are reciprocals only at y values of on the inverse function or through (x, f(x)) and (f^{-1}(f(x)), x). Or the point (3,9) will have a reciprocal slope at (9,3) since at this point x and y are reversed hence the slope becomes the reciprocal or $\frac{\Delta x}{\Delta y}$ This is the important point to understand about the function and its inverse, they only behave as opposites at point (a,b) and (b,a). This means that at point a something different is going on. The question is then how can we find the derivative of the inverse function with respect to the x-axis? Looking again at:

$$y = x^2 \text{ also equals } x = \sqrt{y}$$

$$\frac{\Delta y}{\Delta x} = 2x \quad \Rightarrow \quad \frac{\Delta y}{\Delta x} = 2\sqrt{y} \quad \text{where } x = \sqrt{y}$$

$$\frac{\Delta x}{\Delta y} = \frac{1}{2x} \quad \Rightarrow \quad \frac{\Delta x}{\Delta y} = \frac{1}{2\sqrt{y}}$$

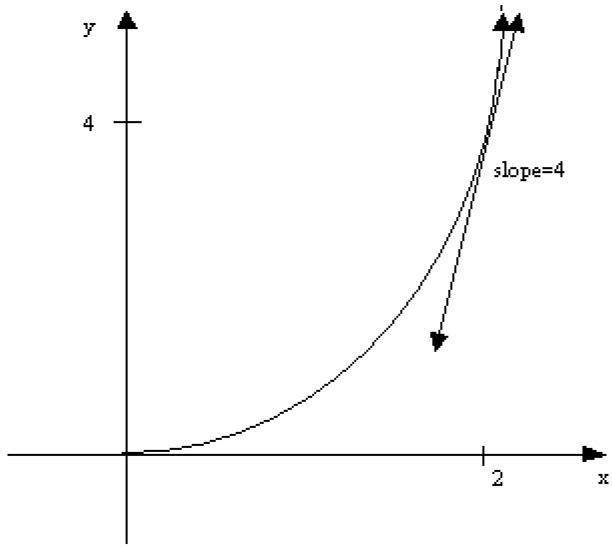
By replacing x with y and y with x in this last expression we get:

$$\frac{\Delta y}{\Delta x} = \frac{1}{2\sqrt{x}}$$

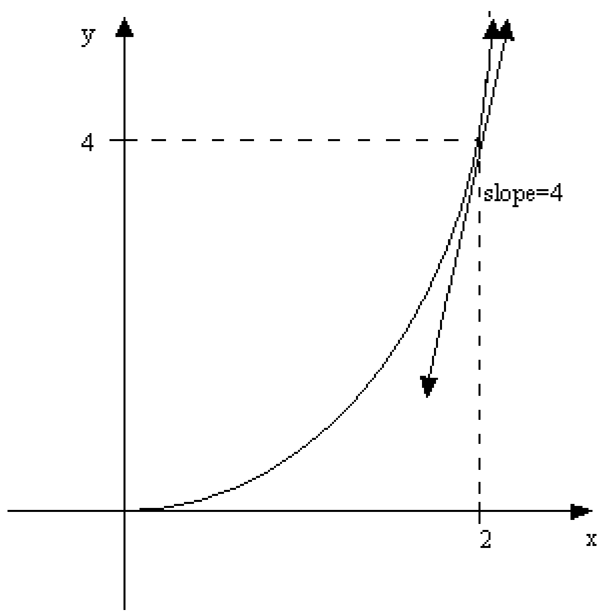
What we have just done is calculated the derivative of the inverse function only by looking at the original function and its derivative. The reason the derivative was not just the reciprocal of y=2x was because we forgot to do the following two steps:

1) Replace x with its equivalent expression in terms of y.

The slope in the following graph is $\frac{\Delta y}{\Delta x}$ at x=2 the slope is (2)(2)=4



By replacing x with $\sqrt{y}$ we can find the derivative with respect to the **same x-axis** but instead with a **y-value.**



At y=4 the slope is $\frac{\Delta y}{\Delta x} = 2x = 2\sqrt{y} = 2\sqrt{4} = \mathbf{4}$ which is the answer we got using x=2 instead.

Since the inverse function is graphed in the same xy plane as $y = x^2$, we can find the derivative of the inverse function with respect to the axis by taking the reciprocal of the expression $\frac{\Delta y}{\Delta x} = 2\sqrt{y}$ and then replacing every y with x and vice-versa.

$$\frac{\Delta x}{\Delta y} = \frac{1}{2\cdot\sqrt{y}}$$

$$\frac{\Delta y}{\Delta x} = \frac{1}{2\cdot\sqrt{x}}$$

This last expression is the derivative for the function.s inverse with respect to the x-axis.

To summarize we can state the following theorem:

To find the derivative of the inverse function,

1) Remember the inverse function is related to the main function by being rotated 90 degrees.

2) First find the derivative of f(x)

3) Replace any x in the derivative with its y-equivalent, so as to be able to find the derivative with any given y-value.

4) Take the reciprocal of the derivative to get $\frac{\Delta x}{\Delta y}$ so as to be able to find the derivative with respect to the y-axis.

5) Since the inverse function is graphed with respect to x, replace every y with x and x with y to find the derivative of the inverse function.

To summarize further:

$$\frac{\Delta y}{\Delta x} \text{ of } f^{-1}(x) = \frac{1}{\frac{\Delta y}{\Delta x} \text{ of } f(x) \text{ ; wherex } = x(y), \text{ then replace all x with y}}$$

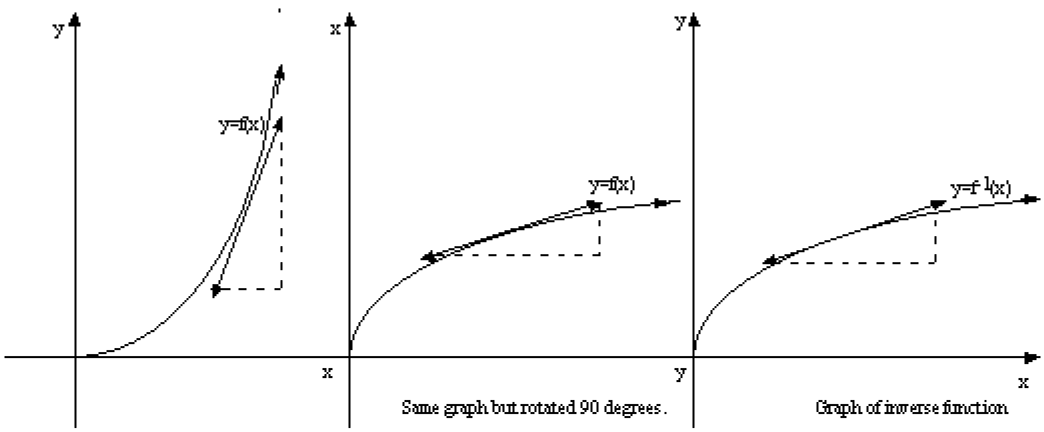
For example given $f(x) = x^3$ find the derivative of its inverse, $f^{-1}(x)$

$$\frac{\Delta y}{\Delta x} = 3x^2$$

$$x(y) = \sqrt[3]{y}$$

$$\frac{\Delta x}{\Delta y} = \frac{1}{3y^{\frac{1}{3}}}$$

$$\frac{\Delta y}{\Delta x} = \frac{1}{3x^{\frac{2}{3}}}$$



Chapter 14 - Introduction to Exponents

This chapter on exponents marks the beginning of the second half of our study of Calculus. In the first half we introduced the concepts of numbers, functions and graphs, then went on to analyze them in more depth using the derivative. Having understood what the derivative was, the remainder of the first half was devoted to integration, or the reverse of differentiation. It required some abstract thinking since what we were doing was analyzing that which we could net directly see.

The first half we basically focused entirely on simple rules of arithmetic. Differentiation was presented as a subtracting and dividing process while Integration was shown to be a multiplicative and additive operation. In other words our study of Calculus was limited to multiplication and addition. From the first chapter you might recall how multiplication was defined as repetitive addition where x multiplied by y was just x plus itself y times. With this idea we can say that all multiplication reduces to addition and this means our study of Calculus was based entirely on addition.

Now we will study Calculus based on multiplication through the study of exponential and Logarithmic function. Before continuing we need to establish a few properties of exponents. The rules that govern exponents are very similar to addition. The definition of an exponent or a power is:

$$a^x$$

Where a is called the base, and x is the exponent or power. Therefore a raised to the x power is defined to be:

$$a^x = a \times a \times a \times a \text{....(a multiplied by itself x times)}$$

For example:

$$5^3 = 5\times5\times5$$
$$7^3 = 7\times7\times7$$
$$8^4 = 8\times8\times8\times8$$

Compare this with the definition of multiplication where:

$$a \times x = a + a + a + a \text{...(a added with itself x times)}$$
$$5\times3 = 5+5+5$$

Since multiplication can be called repetitive addition then exponent or raising something to a power can be though of as repetitive multiplication. This is really all that we mean by exponents. They can be reduced to multiplication and multiplication can be further reduced to exponents.

Now let us take a look at some properties of exponents. The first being the rule for multiplying two exponents of the same base.

$$a^x \times a^y = (a \times a \times a \times a \text{.....x times}) \times (a \times a \times a \times a \text{......y times})$$

$$a^x \times a^y = a^{(x+y)} = (a \times a \times a \times a \text{.....(x + y) times})$$

Notice how multiplication of exponents has been reduced to addition of exponents. For example:

$$10^4 = 10 \times 10 \times 10 \times 10$$

$$10^2 = 10 \times 10$$

$$10^4 \times 10^2 = (10 \times 10 \times 10 \times 10) \times (10 \times 10)$$

$$10^4 \times 10^2 = 10^{(2+4)} = 10 \text{ multiplied by itself 6 times} \quad = 10^6$$

It is really this simple. The second important property of exponents states that a to the x power raised to the y power is just a to the product of x and y power or:

$$\left(a^x\right)^y = a^{(x \times y)}$$

$$\left(a^x\right)^x = (a \times a \times a \text{...x times}) \times (a \times a \times a \text{...x times}) \times (a \times a \times a \text{...x times})\text{... y times}$$

$$\therefore \left(a^x\right)^y = a^{(x \times y)}$$

All this is saying is that an exponent raised to another exponent can be reduced to multiplying the two exponents out or:

$$3^5 = 3 \times 3 \times 3 \times 3 \times 3$$

$$\left(3^5\right)^2 = (3 \times 3 \times 3 \times 3 \times 3)^2 = 3^5 \text{ multiplied by itself two times}$$

$$3^5 \times 3^5 = 3^{(5+5)} = 3^{(5\times 2)} = 3^{10}$$

Take note of how this rule reduces exponents of exponents to just multiplication of the two exponents whereas rule one reduces multiplication of exponents to addition of exponents, (assuming a common base a).

These two important properties of exponents are the fundamental ways of defining what exponents are and how they relate to repetitive multiplication, where multiplication is just repetitive addition. It now remains for us to define what we mean by raising an exponent to a fractional power. This is actually much simpler than it sounds.

Since we defined whole number exponents to denote repetitive multiplication, we would want fractional powers to represent repetitive **division.** Fractional exponents therefore are called roots and tell us into how many equal multiples a number has been divided into ,such that the product of these roots gives us the original number a back.

$$a^{\frac{1}{x}} = \sqrt[x]{a}$$

$$\left(a^{\frac{1}{x}}\right)^{x} = \left(\sqrt[x]{a} \times \sqrt[x]{a} \times \sqrt[x]{a}\text{......x times}\right) = a^{1} = a$$

This definition of fractional exponents remains consistent with the first and second rule of adding and multiplying exponents.

Last but not least we need to define what we mean by raising exponents to negative numbers. This is done by remembering that positive exponents refer to multiplication, therefore negative exponents would refer to division. In order for our definition to remain in accordance with the two rules of exponents we need to define it this way:

$$a^{x} \times a^{-y} = a^{(x+(-y))} = a^{(x-y)}$$

$$a^{x} = a \times a \times a. \times a \times a\text{...x times}$$

$$a^{(x-y)} = a \times a \times a\text{...(x - y) times}$$

$$\left(a^{x} \times a^{-y}\right) \times a^{y} = a^{x}$$

$$a^{x} a^{-y} a^{-y} = a^{x}$$

$$a^{-y} a^{y} = 1$$

$$\mathbf{a}^{-\mathbf{y}} = \frac{\mathbf{1}}{\mathbf{a}^{\mathbf{y}}}$$

This is rather a long way of deriving this as you can clearly see from the following example how negative exponents are defined.

$$6^{5} = 6 \times 6 \times 6 \times 6 \times 6$$

$$6^{5} \times 6^{-2} = 6^{(4-2)} = 6^{3}$$

$$6^{3} = 6 \times 6 \times 6$$

$$6^{-2} \text{ must equal } \frac{1}{6^{2}}$$

$$6^{5} 6^{-2} = \frac{6 \times 6 \times 6 \times \cancel{6} \times \cancel{6}}{\cancel{6} \times \cancel{6}} = 6 \times 6 \times 6 = 6^{3}$$

Increasing the power means multiplying it more times by itself, while decreasing the power means dividing it more times by itself. Therefore multiplication and division reduce to addition and subtraction in exponents.

Since a raised to the first power is a multiplied by itself once, or just a, by definition then what is a raised to the zero power. Once again this definition must remain consistent with the rules of exponents. Instead of raising a to zero power, we can raise it to 1/∞ or one over infinity which is close to zero. Therefore a raised to the negative 1 over infinity is:

$$a^{\frac{1}{\infty}} = \sqrt[\infty]{a}$$

$$a^{\frac{-1}{\infty}} = \frac{1}{\sqrt[\infty]{a}}$$

From the definition of fractional exponents, we are asking our selves what number multiplied by itself an infinite amount of times gives a? If this number were slightly less than 1, then as you multiply infinite times, you get a smaller and smaller number or zero. If this number were slightly greater than 1 then you would get a larger and larger number each time you multiply it by itself, or eventually infinity. Therefore only the number one satisfies both limits as you approach it from either 1/∞ or -1/∞. Hence:

$$a^0 = 1$$

You might think that how can: $\left(a^{\frac{1}{\infty}}\right)^{\infty} = (1)^{\infty} \neq a^1$. This is because zero and 1/∞ are not exactly the same.

This definition of zero exponent power tells us an important property of exponential.s. In multiplication you are adding, where adding nothing is zero. In exponents you are multiplying, where multiplying nothing is one, not zero. Therefore all exponents are expressed as repetitive multiplication of numbers greater than or equal to one. We can say that the base of exponents is therefore 1.

Chapter 15 - Logarithmic and Exponential Functions

The previous chapter was devoted to defining a new type of function, the exponential function. The base of this function was multiplication. We can describe the exponential function as simply:

$$y = a^x \text{ or } f(x) = a^x$$

This is defined for all values of x, as defined in the previous chapter. Let us now look at the inverse of this function or what we shall refer to as the Logarithmic of this function. Remember the inverse function is found by interchanging y and x so we get:

$$x = a^y$$

By solving for y to get the inverse function we get:

$$Somthing(x) = y$$

This tells us that we must do something to an inputted x value to find y. Remember x refers to the number we get by raising a to some power y. Therefore if we were given x then in order to get y, we must ask ourselves to what power has, a, been raised to so as to get y? This question can be replaced with the word . Log., where the Log, base a, of a number is the power a or the base has been raised to so as to get x. Stated mathematically:

$$Log_a x = y$$
$$a^? = x$$

The question marks refers to our output or y values. Remember we are inputting values of x, making x our independent variable, then calculating what power the base must be raised to get x. This answer is our dependent variable as its value depends on what number x we have inputted. Hence:

$$y = f(x) = Log_a(x)$$
$$y = f(1000) = Log_{10}(1000)$$
$$y = 3$$

This is really not that difficult, just remember that the Logarithmic function is the inverse or opposite of the exponential function.

Now let us move on to examine some unique properties of the Logarithm.

$$Log_a cd = Log_a c + Log_a d$$

Or the Log of a product is simply the sum of the Log.s of each number. To understand this, recall the property of exponential multiplication, where the product of two exponential numbers of the same base is simply the sum of the exponents with the same base.

$$a^x a^y = a^{(x+y)}$$
$$10^2 10^3 = 10^5$$

$Log_a cd = Log_a c + Log_a d$ we begin by assuming that c and d can be expressed as some exponent of the base. Remember that $Log_a x = y$ is the same as $a^y = x$. Therefore we can let:

$$c = a^{y1} \text{ where } y_1 = Log_a(c)$$
$$d = a^{y2} \text{ where } y_2 = Log_a(d)$$

This means that $Log_a cd = Log_a (a^{y1})(a^{y2})$ by substituting c and d for a^{y1} and a^{y2} respectively. Then:

$$Log_a (a^{y1} a^{y2}) = Log_a (a^{(y1+y2)})$$
$$Log_a (a^{(y1+y2)}) = y_1 + y_2 = Log_a(c) + Log_a(d)$$
$$\therefore Log_a(cd) = Log_a(c) + Log_a(d)$$

The second important property of Logarithms is that:

$$Log_a\left(\frac{c}{d}\right) = Log_a c - Log_a d$$

The proof is based on property 1, excepts that we must keep in mind that $\frac{a^{y1}}{a^{y2}} = a^{y1} - a^{y2}$. Thus $Log_a\left(\frac{c}{d}\right)$ is expressed as difference of two Logarithms and not a sum, since we are dividing not multiplying.

The final and most important property of Logarithms is that:

$$Log_a(x)^n = nLog_a(x)$$

The proof of this extremely useful property requires us to first re-write $Log_a(x)$ as $a^y = x$ where $y = Log_a(x)$. When we raise x to the n we must raise the left side of the equation also to the n.th power:

$$\left(a^y\right)^n = x^n$$
$$a^{yn} = x^n$$

This is based on the third property of exponential functions that tells is that an exponents raised to an exponent can be reduced to the product of the exponents. We can now write $a^{yn} = x^n$ as $a^{ny} = x^n$ which is the same as

$$Log_a x^n = ny \text{ where } y = Log_a(x)$$
$$\therefore\ Log_a x^n = nLog_a(x)$$

Having proved the unique propertied of Logarithms, all of which related to the property of exponents, we shall now calculate the derivative of the logarithmic function. We begin with the function:

$$y = Log_a(x)$$

From the definition of the derivative:

$$\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Using this we get:

$$\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{Log_a(x+\Delta x) - Log_a(x)}{\Delta x}$$

From the second property of Logarithms the numerator: $Log_a(x+\Delta x) - Log_a(x)$

Can be simplified to:

$$Log_a\left(\frac{x+\Delta x}{x}\right) = Log_a\left(1+\frac{\Delta x}{x}\right)$$

Substituting this back into the definition of the derivative we get:

$$\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{Log_a\left(1+\frac{\Delta x}{x}\right)}{\Delta x} \text{ or } \frac{1}{\Delta x} Log_a\left(1+\frac{\Delta x}{x}\right)$$

Recall the third property of Logarithms; $Log_a x^n = nLog_a(x)$ Consequently: $\frac{1}{\Delta x} Log_a\left(1+\frac{\Delta x}{x}\right)$ also equals $\frac{1}{\Delta x} Log_a\left(1+\frac{\Delta x}{x}\right)^{\frac{1}{\Delta x}}$

Clearly as Δx goes to zero, Δx / x also goes to zero regardless of what value x is. Outside the brackets, 1/ Δx tends to infinity as Δx goes to zero. This means we can rewrite:

$$Log_a\left(1+\frac{\Delta x}{x}\right)^{\frac{1}{\Delta x}} \text{ as } Log_a\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}\cdot\frac{1}{x}}$$

x / Δx is therefore the reciprocal of Δx / x and it also tends to infinity as Δx approaches zero, regardless of what value x takes on. Thus from the third property of Logarithms:

$$Log_a\left(1+\frac{\Delta x}{x}\right)^{\frac{1}{\Delta x}} Log_a\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}\cdot\frac{1}{x}} = \frac{1}{x} Log_a\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}$$

We shall now make a small change in variables to help evaluate the limit inside the Log brackets. We replace Δx / x with 1/n and x / Δx with n. The derivative can now be re-written as:

$$\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0} \frac{1}{x} Log_a\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}$$

which now becomes

$$\frac{1}{x}\lim_{n\to\infty} Log_a\left(1+\frac{1}{n}\right)^{n}$$

Let us now evaluate the limit an n -> infinity of $\left(1+\frac{1}{n}\right)^{n}$. By the Binomial Theorem we get:

$$\left(1+\frac{1}{n}\right)^{n} = 1+n\left(\frac{1}{n}\right)+\frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^{2}+\frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^{3}+\frac{n(n-1)(n-2)..........(n-(n-1))}{n!}\left(\frac{1}{n}\right)^{n}$$

$$\left(1+\frac{1}{n}\right)^{n} = 1+1+\frac{\left(1-\frac{1}{n}\right)}{2!}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!}+.........\frac{\left(1-\frac{1}{n}\right).....\left(1-\frac{(n-1)}{n}\right)}{n!}$$

$$\left(1+\frac{1}{n}\right)^{n} = 2+.......$$

The sum of the terms after the first two will always be less than the sum of the corresponding infinite series $\sum_{n=1}^{\infty}\frac{1}{2^n}$ whose sum can be proven to be one. Just think that it will take an infinite number of steps to cover one meter if in each step you can only cover half the remaining distance.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+.......\frac{1}{2^{n-1}}$$

This is true because: $\frac{\left(1-\frac{1}{n}\right)}{2!} < \frac{1}{2^1}$ or $\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!} < \frac{1}{2^2}$. The same reasoning can be applied to the remaining n terms; the numerator will always be less than one and the denominator will be greater than the denominator of the corresponding n.th term in the series. For example:

$$\frac{1}{n!} < \frac{1}{2^{n-1}} \text{ or } n! > 2^{n-1}$$
$$5\times 4\times 3\times \cancel{2}\times \cancel{1} > 2\times 2\times 2\times \cancel{2}\times (\cancel{1})$$
$$5\times 4\times 3 > 2\times 2\times 2$$

Since the sum of this series is 1, we can make an important conclusion,

$$2 < \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n < 2+1 = 3$$

As we shall see later, this limit tends to the number e, where e=2.71828...Returning back to our derivative we get.

$$\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{1}{x}Log_a\left(1+\frac{1}{-}\right)^{\frac{x}{\Delta x}}$$

Therefore:

$$\frac{\Delta y}{\Delta x} = \frac{1}{x}Log_a e$$

If the base a was set equal to e, we would have:

$$\frac{\Delta y}{\Delta x} = \frac{1}{x}Log_e e$$
$$\frac{\Delta y}{\Delta x} = \frac{1}{x}\cdot 1\ (e^1 = e)$$
$$\frac{\Delta y}{\Delta x} = \frac{1}{x}$$

For this reason we denote the Logarithm of base, e, to be called the Natural Logarithm or ln x , whose derivative is simply 1/x .The derivative of logarithms of other bases is $\frac{\Delta y}{\Delta x} = \frac{1}{x}Log_a e$, where the Log e is some other constant other than 1.

Having shown that the derivative of ln x is 1/x we can go on to prove the derivative of the inverse of the logarithm function, the exponential function, using our algorithm presented in the chapter on inverse functions. Before doing it that way let us use the definition of the derivative to find the exponential function.s derivative.

To do this we follow similar steps to those we used to calculate the derivative of the Logarithm of base a, function. Instead of $a^y = x$ **we have** $a^x = y$

$$\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{a^{(x+\Delta x)} - a^x}{\Delta x}$$

Recall the first property of exponential multiplication, where $a^x \cdot a^y = a^{(x+y)}$ or $10^3 10^9 = 10^{12}$

$$\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{a^x a^{\Delta x} - a^x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{a^x (a^{\Delta x} - 1)}{\Delta x} \quad \text{(Factoring out } a^x\text{)}$$

Therefore $a^{(x+\Delta x)} = a^x a^{\Delta x}$ and

$$= \lim_{\Delta x \to 0} a^x \left(a^{\Delta x} - 1\right) \frac{1}{\Delta x}$$

As Δx gets closer and closer to zero,

$a^{\Delta x} \approx a^0 = 1$ and $\frac{1}{\Delta x}$ tends to infinity

We will now make a small change in variable to simplify the limit calculation. We replace Δx with $\frac{1}{n}$ as $n \to \infty$. We now have: $\frac{\Delta y}{\Delta x} \lim_{n \to \infty} a^x \left\{\left(a^{\frac{1}{n}} - 1\right)n\right\}$

Let us now evaluate the limit in the brackets.

$$\lim_{n \to \infty} \left(a^{\frac{1}{n}} - 1\right)n$$

$$\lim_{n \to \infty} \left(\sqrt[n]{a} - 1\right)n$$

The value within the parenthesis will tend to zero or $\frac{1}{n}$ as $n \to \infty$ since for example $1.0001 - 1 = \frac{1}{10000}$ where n ≈ 10000. We now want to solve $\left(\sqrt[n]{a} - 1\right)$ for $\frac{1}{n}$ $\left(\frac{1}{\cancel{n}}\right)\cancel{n} = 1$ and the derivative will be a^x

Solving the part in parenthesis gives us:

$$\sqrt[n]{a} - 1 = \frac{1}{n}$$

$$\sqrt[n]{a} = 1 + \frac{1}{n}$$

$$a^{\frac{1}{n}} = 1 + \frac{1}{n}$$

Raising both sides to the n.th power, we get:

$$\left(a^{\frac{1}{n}}\right)^n = \left(1+\frac{1}{n}\right)^n$$

$$a = \left(1+\frac{1}{n}\right)^n$$

Remember we are taking the limit as n goes to infinity so we have:

$$a = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \mathbf{e}$$

This tell us that if the base a were equal to e the derivative of a^x

$$\frac{\Delta y}{\Delta x}\lim_{n\to\infty} a^x \left\{\left(a^{\frac{1}{n}}-1\right)n\right\}$$

Becomes:

$$\frac{\Delta y}{\Delta x} = e^x(1) = e^x$$

The reason we get 1 is because:

$$\lim_{\Delta x\to 0}\left(a^{\Delta x}-1\right)\frac{1}{\Delta x} \text{ is the same as } \lim_{n\to\infty}\left(a^{\frac{1}{n}}-1\right)n$$

Setting a equal to e gives us:

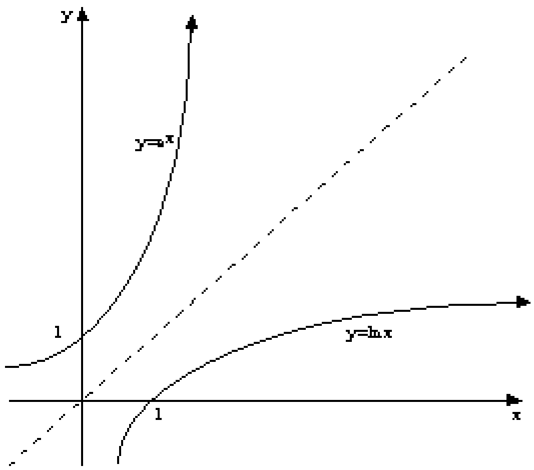
$$\frac{\Delta y}{\Delta x}\lim_{n\to\infty} e^x \left\{\left(e^{\frac{1}{n}}-1\right)n\right\} \text{where } \left(e^{\frac{1}{n}}-1\right)n \text{ becomes :}$$

$$\left(\sqrt[n]{\left(1+\frac{1}{n}\right)^n}-1\right)n = \left(\left(1+\frac{1}{n}\right)-1\right)n = \left(\frac{1}{n}\right)n = 1$$

$$\therefore \frac{\Delta y}{\Delta x} = e^x$$

Remember this only happens when a=e. The derivative of $y = e^x$ is e^x or the function itself. This tells us that the rate of change of the function at a point x is proportional to the function.s value at that point. This will be discussed further in the next chapter.

Now let us examine the graphs of the Natural Logarithm function, $y = \ln x$, and the exponential function of base e, $y = e^x$. Remember these two functions are inverses of each other, since only the dependent and independent variable have switched places, or x and y.



Since we know that the derivative of y = ln x is simply 1/x, we can then find the derivative of the inverse function, the exponential function, with respect to the x-axis, using our algorithm presented in the chapter on inverse functions. It goes as follows:

$$y = \ln_e x \text{ or } e^y = x$$

$$\frac{\Delta y}{\Delta x} = \frac{1}{x} \text{ where } x = e^y$$

$$\frac{\Delta y}{\Delta x} = \frac{1}{e^y}$$

Taking the reciprocal :

$$\frac{\Delta x}{\Delta y} = e^y$$

Replacing x with y :

$$\frac{\Delta \mathbf{y}}{\Delta \mathbf{x}} = \mathbf{e}^x$$

We have just found proven that the derivative of $y = e^x$ is the function itself or e^x y. = y. This was done by just examining the function.s inverse, the natural logarithm and its derivative with respect to x. This shows the close relationship the Exponential Function has with the Logarithmic function, the two being inverses of one another. This concludes our study of these two function.s and their derivatives. We will explain later how e is actually a number defined as a limit. For now just consider how in calculating the derivative of the multiplicative function, the exponential function, involved an analysis of each term in the series of the binomial expansion, whereas in the study of repetitive addition functions, polynomials, all the terms except the second, went to zero.

Chapter 16 - Applications of the Exponential Function

In this chapter we will take a look at just a few of the many interesting situations where the exponential function arises. The emphasis on the exponential function was that its base was multiplication, or it can be thought of as a repetitive multiplication function.

We proved its derivative, but now we need to explain how to integrate it. You may recall from Part I, that in integrating a function over any interval, we just added a constant to our integral. The derivative of the function with or without an added constant was still the same. The important point to realize here is that if we were calculating the integral over a definite interval from a to b, then the constants are useless and take no part in the calculation. For example if the integral of f(x) is F(x) + C, then the integral from a to b is:

$$\int_a^b f(x)\Delta x = \Big[_a^b F(x) + C = \big(F(a) + C\big) - \big(F(b) + C\big) = F(a) + \cancel{C} - F(b) - \cancel{C} = F(a) - F(b)$$

This tells us that the change in a function over an interval is independent of the constants that are added on. The constant are there to define the initial condition, such that for any, x, F(x) is simply plus or minus a constant. This will become clearer later on as we look at examples.

If constants are added on to polynomial functions, then they are multiplied on to exponential functions. What do we mean by this? When we are integrating the Logarithmic and exponential functions we can multiply constants that have no effect on the derivative, they are there only to satisfy initial condition. For example, the integral of 1/x is:

$$\int \frac{1}{x}\Delta x = \ln x$$

What is then the derivative of ln of a constant times x?

$$y = \ln cx$$
$$\frac{\Delta y}{\Delta x} = \frac{1}{\cancel{c}x}\cdot\cancel{c} = \frac{1}{x}$$
$$\therefore \int \frac{1}{x}\Delta x = \ln cx$$

Note that we have not attached any limits to the integral, this is called the indefinite integral as it simply refers to any x over any interval. Suppose we were to integrate over a definite interval from a to b, then just as the constants in the polynomial functions canceled out, so will the constant in ln cx:

$$\int_a^b \frac{1}{x}\Delta x = \Big[_a^b \ln cx = \ln bc - \ln ac = \ln\frac{b\cancel{c}}{a\cancel{c}} = \ln b - \ln a$$

Therefore the constant plays no role in the change of function over an interval and is only there to define the initial condition..

Now let us move on to some applications, first a look at populations. Population growth is generally expressed as a percentage of the population per year. Such that a 3% growth rate means that a population of 1000 people will see 3 new babies for the first year, 3% of 1003 for the next year and so on. No matter how large the population, there will always be 3% of the population wanting to have babies every year. We can express this relationship as follows:

$$\frac{\Delta Population}{\Delta time} \quad \text{or} \quad \frac{\Delta p}{\Delta t} = .03p \ \text{ or 3\% of the population}$$

This can be written more generally as:

$$\frac{\Delta P}{\Delta t} = r\cdot P \ \text{ (r = Growth rate per unit time)}$$

The only function whose rate of change or derivative is proportional to itself is the exponential function or:

$$y = e^x$$
$$\frac{\Delta y}{\Delta x} = e^x = y$$

A more accurate way of solving this, is separating the variables and integrating the change in population from some initial to some final population, and integrating time from 0 to t.

$$\frac{\Delta P}{\Delta t} = rP$$
$$\Delta P = rP\Delta t$$
$$\frac{\Delta P}{P} = r\Delta t$$

We can now integrate both sides over the limits just defined:

$$\int_{P_{present}}^{P_{future}} \frac{1}{P}\Delta P = \int_0^t r\Delta t$$
$$\ln P_{final} - \ln P_{initial} = r \cdot t$$
$$\ln_e \frac{P_{final}}{P_{initial}} = r \cdot t$$

From the definition of the Logarithm we get:

$$\frac{P_{final}}{P_{initial}} = e^{rt}$$
$$\mathbf{P_{final} = P_{initial}e^{rt}}$$

Note how the initial condition of a given population at t=0 is multiplied on to the exponential function and not added on, such that at t=0 we have:

$$P_{final} = P_{initial}e^{(r)(0)}$$
$$P_{final} = P_{initial}e^{0} \quad (e^{o} = 1)$$
$$P_{final} = P_{initial} \quad \text{at } \mathrm{t} = 0.$$

Consider the population of Pakistan, 120,000,000 people. If the birth rate is 3% per year, then what will be the population in 10 years?

We do not take the death rate into consideration is because the birth rate, takes into account both the birth and death rate such that it reflects the net or true growth in population per year. Our equation becomes:

$$\text{Future Population} = \text{Present Population } (e)^{birthrate \cdot time}$$

$$P = P_0 e^{rt}$$

$$P = 110{,}000{,}000 e^{(.04)(10)}$$

$$P = 110{,}000{,}000(2.71)^3$$

$$P = 148{,}484{,}468$$

The population will therefore be over 148,484,468 people which represents an increase of 38,484,468 people in ten years. Note that this answer was **not** found by taking 3% of 110 million and multiplying by ten. This fails to work since it assumes that every year the population is 110 million, but this is clearly not the case, since the population is constantly growing and growing and 3% of the population is a larger and larger number. That is why 38,484,468 > 33,000,000.

If you are a skeptical person, like me, you will probably doubt how accurate our answer is. One argument might be that since the population is increasing every second and not only once a year, then 38,484,468 is not the exact increase as it is based on a yearly increase in population growth. In fact our answer does indeed take into consideration that babies are born every second and not only one a year. To understand this, go back to our proof of the derivative of the exponential function:

$$\frac{\Delta P}{\Delta t} = \lim_{\Delta t \to 0} P_o e^{rt}$$

We are taking the limit as time goes to zero and similarly the integral:

$$\frac{\Delta P}{\Delta t} = r \cdot P$$

$$\int \frac{\Delta P}{P} = \int r \Delta t$$

Is also based on Δt going to zero. Furthermore the growth rate is a constant value and will always be 3%, in this case, regardless of how much time has passed in years, months or seconds. The fact remains that there will always be 3% of the population desiring to have babies.

$P = P_0 e^{rt}$ is based on time going to zero, how can we be sure that when we solve for t =10, it is referring to ten years and not ten months or ten seconds later.

The way we are sure that 10 refers to years and not anything else, is by remembering that 3% growth rate is a per year figure. So.

$$\frac{\Delta P}{\Delta t} = .03P$$

means that even though the growth rate is always 3%, it doesn't take into consideration that it takes around a year for a baby to be born, so that 3% per year means that the there will be .03P new babies per year, not per month.

Another fault with our equation is that it assumes that the 3 percent of the entire population at any given time t, has an equal desire to have a baby in one year. This is obviously untrue. Therefore our population equation is clearly flawed but the general idea is correct.

The continent of South Asia has an average growth rate of 2.5% per year. If the current population is 1.1

billion, in how many years will the population double or be 2.2 billion. Our equation:

$$\text{Future Population} = \text{Present Population}\ e^{rt}$$

$$2.2\,billion = 1.1billion \cdot e^{.025t}$$

$$2 = e^{.025t}$$

Taking the natural log of both sides gives us:

$$\ln 2 = .025t.$$

$$.693147 = .025t$$

$$t = 27.7$$

In 27 years the Population of South Asia will double.

In this next example we will take a closer look at the exponential function's role as a solution to the equation: $y' = ry$. Hopefully all doubts will be cleared up now.

When one deposits money in a savings account, it is generally for letting the money earn interest. Interest is just money paid for the use of money. Interest rates are quoted as per year or 6% per year or 9% per annum (per year). So a 6% rate would mean that $1000 would earn $\frac{6}{100}1000 = \$60$. This is called simple interest since the money only earn interest once a year.

Compound interest on the other hand, is where the interest earns interest. So $10,000 put into an account that earns 6% interest, compounded quarterly or every three months, undergoes a series of calculation. In each quarter we add the interest earned from the previous quarter to the amount of money in the bank to get our new balance for the next quarter. Therefore the interest earned in the previous quarters adds to the interest earned in the following quarters

Quarter 1:

$$Interest = \Pr incipal \times Rate \times Time$$

$$Interest = (10,000) \times \left(\frac{6}{100}\right) \times \left(\frac{1}{4}\right) = \$150$$

Quarter 2:

$$Interest = (10,000 + \text{ Earned Interest }) \times \left(\frac{6}{100}\right) \times \left(\frac{1}{4}\right)$$

$$Interest = (10,000 + \ 150) \times \left(\frac{6}{100}\right) \times \left(\frac{1}{4}\right) = \$152.25$$

Quarter 3:

$$Interest = (10{,}150 + 152.25) \times \left(\frac{6}{100}\right) \times \left(\frac{1}{4}\right) = \$154.53$$

End of 1 Year:

$Interest = (10,302.25 + \ 154.53) \times \left(\frac{6}{100}\right) \times \left(\frac{1}{4}\right) = \$156.85$

$Total = \$10613.63 > \$10600$

The difference is not much since the amount of interest earning interest is small. The balance in the bank after one period for money compounded n times a year can be found by:

$$\Pr incipal_{final} = \Pr incipal_{initial} + Interest$$

$$P_{final} = P_{initial} + P_{initial}(rate)(time) \text{ The time is } \frac{1}{n} \text{ for money compounded n times a year}$$

$$P_{final} = P_{initial} + P_{initial}(r)\left(\frac{1}{n}\right)$$

$$P_{final} = P_{initial}\left(1 + \frac{r}{n}\right)$$

This last formula gives us the new balance after the first time it was compounded. To find the amount for the second compounding period we multiply by:

$$P = P_{before} + Interest$$

$$P = P_{initial}\left(1 + \frac{r}{n}\right) + P_{initial}(rate)(time) \ t = \frac{1}{n}$$

$$P = P_{initial}\left(1 + \frac{r}{n}\right) + P_{initial}\left(1 + \frac{r}{n}\right)\left(\frac{r}{n}\right)$$

Factoring out a $P_{initial}\left(1 + \frac{r}{n}\right)$ gives us:

$$P = P_{initial}\left(1 + \frac{r}{n}\right)\left(1 + \frac{r}{n}\right)$$

$$P = P_{initial}\left(1 + \frac{r}{n}\right)^2$$

The money in the bank after the 3.rd and 4.th it has been compounded is given by $P_{initial}\left(1 + \frac{r}{n}\right)^3$, $P_{initial}\left(1 + \frac{r}{n}\right)^4$. By observing the pattern we can conclude that the balance at the end of the year, when the money has been compounded n times is:

$$\mathbf{P_{initial}\left(1 + \frac{r}{n}\right)^n}$$

If $10,000 were put into a savings account that gives an interest rate of 6%. compounded quarterly,

$$P_{final} = P_{initial}\left(1+\frac{r}{n}\right)^{n}$$

$$P_{final} = 10,000\left(1+\frac{.06}{4}\right)^{4}$$

$$P_{final} = \$10,613.63$$

If this new balance were put into the same account for another year we would have:

$$P = P_{initial}\left(1+\frac{r}{n}\right)^{n}$$

$$P = 10,000\left(1+\frac{.06}{4}\right)^{4}\left(1+\frac{.06}{4}\right)^{4}$$

$$P = 10,000\left(1+\frac{.06}{4}\right)^{\mathbf{4\times 2\ or\ 4+4}}$$

$$P_{\text{after two years}} = 10,000\left(1+\frac{.06}{4}\right)^{8}$$

The formula then for calculating the new balance, or how much money is in the bank after t years compounded n times is:

$$\Pr incipal = P = P_{initial}\left(1+\frac{r}{n}\right)^{(nt)}$$

The exponent (nt) refers to how many times the money has been compounded. If in one year it is compounded n times, then in t years it is compounded nt times over the t years.

If the same money were compounded infinite times this would mean that the interest is constantly earning interest. The interest earned over a small Δt is added to the previous balance, creating a new balance on which the interest earned over the next Δt is based on. Therefore the rate at which the money is changing is proportional to the amount of money in the bank times the interest rate.

$$\Pr incipal = P_0 \cdot \lim_{n\to\infty}\left(\left(1+\frac{r}{n}\right)^{n}\right)^{t}$$

Let us examine this familiar looking limit in the brackets, $\left(1+\frac{r}{n}\right)^{n}$. It looks similar to the definition for e, except for the r, interest rate which is constant throughout the time interval. To evaluate this recall that the limit as Δx goes to zero of $x/\Delta x$ and $\Delta x/x$ were infinity and zero, respectively. This is true regardless of the value of x, since its value was fixed and did not change. Therefore by raising $\left(1+\frac{r}{n}\right)^{n}$ to the 1/r power gives us:

$$\lim_{n\to\infty}\left(\left(1+\frac{r}{n}\right)^{n}\right)^{\frac{1}{r}} = \left(1+\frac{r}{n}\right)^{n\cdot\frac{1}{r}} = \left(1+\frac{r}{n}\right)^{\frac{n}{r}}$$

Now as n tends to infinity, r/n goes to zero, and n/r goes to infinity. This gives us the familiar answer, e.

$$\therefore \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{\frac{n}{r}} = \mathbf{e}$$

Raising both sides to the, r, power we get:

$$\lim_{n\to\infty}\left(\left(1+\frac{r}{n}\right)^{\frac{n}{r}}\right)^{r} = e^{r}$$

This gives us the limit of what we were initially looking for:

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{\frac{n}{r}\cdot r} = \left(1+\frac{r}{n}\right)^{n} = e^{r}$$

Or more generally for any x,

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n} = e^{x}$$

Returning back to our initial problem of calculating the Principal in an account that compounds interest infinite times per year:

$$\Pr incipal = P_0 \lim_{n\to\infty}\left(\left(1+\frac{r}{n}\right)^{\frac{n}{r}}\right)^{t}$$

$$\Pr incipal = P_0(e^{r})^{t}$$

$$\mathbf{Principal} = \mathbf{P_0} \cdot \mathbf{e^{rt}}$$

This important answer could also have been arrived at by realizing that money that is put into an account that compounds interest infinitely is changing at a rate proportional to the amount of money in the bank, times the interest rate. For example if at t=4, the amount of money in the bank was $10000, and the interest rate was 6%, then at that instant the money would be changing at the rate of $600 a year. As the money changes then so does the amount of money in the bank, such that our equation becomes:

$$\frac{\Delta Money}{\Delta time} = rate \cdot Money$$

We can call Money, Principal or just P, and separate variables:

$$\frac{\Delta P}{P} = r\Delta t$$

We can integrate this equation to solve for P in terms of r and t. We integrate time from 0 to t, and Principal, from P at t=0, to some final value of P.

$$\int_{P_0}^{P} \frac{\Delta P}{P} = \int_{0}^{t} r\Delta t$$

This yields the solution:

$$\ln P - \ln P_0 = rt$$

$$\ln_e \frac{P}{P_0} = rt$$

$$e^{rt} = \frac{P}{P_0}$$

$$P_0 e^{rt} = P$$

$$\therefore P = P_0 e^{rt}$$

This last answer is the same answer we derived using the definition of compound interest, which gives us the amount of money in a bank after t years and infinite compounding with an initial deposit of P_0. Differentiating this equation with respect to time:

$$P = P_o e^{rt}$$

$$\frac{\Delta P}{\Delta t} = r\left(P_o e^{rt}\right)$$

$$\frac{\Delta P}{\Delta t} = rP$$

Despite how difficult it may seem to explain this change the fact remains that the money is changing at a rate proportional to the amount of money itself, such that as the money begins to change over a small change in time, the amount of money in the bank changes by a small amount creating a new balance that is only accurate for the next Δt . It is really this simple.

In this example we will compare simple interest to compound interest. Consider a poor college student who decides to put his summer earnings of $2000 into a savings account for 3 years. His choices are to Watchovia Bank that offers a simply interest rate of 3.2% per year or Carolina Bank that offer an interest rate of 2.9% compounded continuously. Where should he put his money? Also note that Watchovia offers a free T-shirt for opening a savings account, that makes an excellent Pajama top.

Lets first consider Watchovia. The rate at which his money will be changing is the rate of interest time the amount of money in the bank or (3.2/100)(2000) = $64/year. Remember this is the definition of simple interest, the rate at which your money is changing is only dependent on how much Money you initially open the savings account with.

$$\frac{\Delta M}{\Delta t} = \$64 / year$$

$$\Delta M = 64 \Delta t$$

$$\int_{2000}^{M} \Delta M = \int_{0}^{3} 64 \Delta t$$

$$M - 2000 = 64(3)$$

$$M = 2000 + 192 = \mathbf{\$2192}$$

With Carolina Bank, the interest rate is 2.9% compounded continuously. We can just go directly to our equation for continuous compounding:

$$P = P_0 \cdot e^{rt}$$

$$P = 2000 \cdot e^{032(3)}$$

$$P = 2000 \cdot e^{096}$$

$$\mathbf{P = \$2201.52}$$

This tells us that the interest earning interest only contributed to an extra $9.52, barely more than the cost of the shirt. This is because the interest earned was small, and the interest it earns is even smaller. Now if we were to put the money on for 30 years, then the interest earned will have enough time to earn some substantial interest.

$$Wachoviain 30 years = 2000 + (64)(30) = \$3920$$

$$Carolinain 30 years = 2000 \cdot e^{.032(30)} = \$5223.39$$

In simple interest the amount of money in the bank remains constant, while in compound interest the balance is increasing constantly.

Just as exponential functions increase with rates proportional to the function itself, they can also decrease. This happens when:

$$\frac{\Delta y}{\Delta x} = -ky$$

Whose solution is simply:

$$y = y_0 \cdot e^{-kt}$$

Consider the element Uranium that undergoes radioactive decay at the rate of 12% of the material per year. This decay rate, 12%, is independent of the amount of material present. Find the time taken for half of Uranium to disappear, or the half life.

$$\frac{\Delta U}{\Delta t} = .12U$$

$$U = U_0 \cdot e^{-.12t}$$

$$\frac{U}{U_0} = e^{-.12t}$$

$$\frac{1}{2} = e^{-.12t}$$

$$\ln .5 = .12t$$

$$t = 5770 \text{ years}$$

This half life is true for any amount of material. A 1000 kg sample and a 1 kg sample will both take the same time to reduce to half their original mass.

This concludes our study of the exponential and logarithmic function and their functions. It is important to understand the differences and similarities to polynomial function. The base of these two functions is multiplication and exponential growth can often be though of as repetitive multiplication. In this chapter we saw how this applies to multiplying populations and multiplying money. Exponential decay occurs whenever the rate at which the function is decreasing is proportional to the function itself.