CHAPTER 17 - The Sine and Cosine Function

I see crowds of People, walking around in a Ring. T.S. Eliot Wasteland

A current under sea Picked his bones in Whisper. As he rose and fell
He passed the stages of his age and youth
Entering the whirlpool. Wasteland

Your memory of trigonometry is probably filled with meaningless identities and strange acronyms to help remember what all the Trig. Functions refer to. Some of you may recall that infamous Indian Tribal Chief, SohCahToa, whose name helped you pass your equally notorious quizzes and tests. (Sine opposite over adjacent, Cosine adjacent over hypotenuse, Tangent, opposite over adjacent; hence SohCahToa) Others may remember being presented with a circle, called the Unit Circle for reasons soon to be seen. In this circle any point on it had the co-ordinates (Cos Ø, Sin Ø) where x = Cos Ø and y = Sin Ø. Often times the circle added more confusion to your understanding of Sines and Cosines. At such a point your mind must have justifiably asked, . What do Indian Tribal Chiefs and Circles have to do with the study of the Sine function, a function whose application arises everywhere from the ten second swaying period of the World Trade Center towers to the mechanisms that translate the back and forth motions of the watch spring into time. In almost any study of oscillating or vibrating systems, the Sine function arises, for reasons that are not attributed to mere coincidence and chance.

In all simplicity the Sine function is the complex mathematical function to describe circles. From the lines from T.S Eliots Wasteland and from your own experience circles represent a never ending, repetitive cycle. Motion along a circle passes through the same point infinite times. It is this complex nature of the circle, that causes its study to be so intense and intricate. Before we can talk about circle, we need to first understand triangles.

Let us begin our study of the Sine function with a look at right triangles. In all simplicity the Sine of an angle (The issue of what is an angle and how to describe it will be dealt with later) is the ratio of the opposite side to the hypotenuse:

\[
\sin (\theta) = \frac{BC}{AC} \quad \text{where } AC \text{ and } BC \text{ refer to the hypoteneuse and opposite side respectively}
\]

Or more directly:

\[
\sin (\theta) = \frac{2}{5.4} \\
\sin (\theta) = .37
\]

From the calculator or a Trig. Table we can find the angle \( \theta \) approximately. The Cosine functions similar to the
Sine function except that it measures the adjacent side, not the opposite side, ratio to the hypotenuse. For example:

In right triangle ABC,

\[ \cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{6}{10} \]

\[ \cos(\theta) = 0.6 \quad \theta = 53.1^\circ \]

The last trig function, the Tangent, is the ratio of the opposite side to the adjacent side. Thus in right-triangle ABC,

\[ \tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{8}{6} \]

\[ \tan(\theta) = 1.33 \quad \theta = 53.1^\circ \text{ From our Trig table.} \]

The remaining Trigonometric functions; Secant, Cosecant, and Cotangent are simply the reciprocal values of the Cosine, Sine, and Tangent respectively.

\[ \sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}} \quad \text{Secant}(\theta) = \frac{\text{Hypotenuse}}{\text{Opposite}} \]

\[ \cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}} \quad \text{Cosecant}(\theta) = \frac{\text{Hypotenuse}}{\text{Adjacent}} \]

\[ \tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}} \quad \text{Cotangent}(\theta) = \frac{\text{Adjacent}}{\text{Opposite}} \]

It is not important to remember the reciprocal functions as they unnecessarily add confusion to one's understanding of Sine, Cosine, and Tangent. It is enough to remember that they are just 1 over the value for the Sine, Cosine or Tangent.

To further make life easier one can express Tan (\( \theta \)) as just Sin (\( \theta \)) over Cos (\( \theta \)). This is easy to see as:

\[ \tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}} \quad \text{then} \quad \frac{\sin(\theta)}{\cos(\theta)} = \frac{\frac{\text{Opposite}}{\text{Hypotenuse}}}{\frac{\text{Adjacent}}{\text{Hypotenuse}}} = \frac{\text{Opposite}}{\text{Hypotenuse}} \cdot \frac{\text{Hypotenuse}}{\text{Adjacent}} = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{\text{Opposite}}{\text{Adjacent}} \]
This reduces the study of Trigonometry to only two functions, the \( \text{Sin}(\theta) \) and \( \text{Cos}(\theta) \). Any other Trig function can be expressed as a variation of these two. To state them again they are:

\[
\text{Sin}(\theta), \quad \frac{1}{\text{Sin}(\theta)}, \quad \text{Cos}(\theta), \quad \frac{1}{\text{Cos}(\theta)}, \quad \frac{\text{Sin}(\theta)}{\text{Cos}(\theta)}, \quad \text{and} \quad \frac{\text{Cos}(\theta)}{\text{Sin}(\theta)}
\]

At this stage your mind should only be focused on knowing what the Sine and Cosine of an angle mean. Sine is opposite over hypotenuse, and Cosine is adjacent over hypotenuse.

Now there are two very important concepts to understand about the Sine and Cosine of a given angle. First, Sines and Cosines apply only to **right triangles**, when there is one right angle of 90 degrees, whose opposite side is called the hypotenuse. If one were asked to find the Sine of the following angle.

\[ \theta \]

Then the answer is not 2.8/5.5 or 2.8/7 as triangle ABC is clearly not a right triangle and side AC is not a hypotenuse. To find \( \text{Sin}(\theta) \) we would have to somehow construct a right triangle, probably the easiest way by drawing an altitude from C perpendicular to AB, like this:

![Diagram of triangle with altitude drawn](image)

The values for \( h \) and \( x \) can be found by writing two simultaneous equations based on pythagorean's theorem:

\[
\begin{align*}
    x^2 + h^2 &= (2.8)^2 \\
    (7-x)^2 + h^2 &= (5.5)^2
\end{align*}
\]

The second important point to understand about Sines and Cosines of angles is that their values are independent of the dimensions of the triangle. What this means is that the Sine of a 62 degree angle will always be .883, regardless of the size of triangle it is measured in.
Size is a meaningless quantity unless it is used with reference to something else. If you examine a right triangle individually then you will find it impossible to describe its size. It is only when you look at it with respect or relative to another triangle that you can correctly say the following concerning these two right triangle.

Triangle A is large. (Relative to B only)

Now if we were to compare Triangle A with let us say another Triangle C we would say:

Triangle A is small. (relative to C)

Finally we can compare Triangle A with two other triangles and say the following:
Triangle A is of average size. (relative to b and C).

This brings us to a logical question; how can Triangle A be small, large, and even average? Since Triangle A is unique and does not change then it is our adjectives which are at fault in describing the attribute of size of the triangle. We are now in a position to state an important philosophical idea: Adjectives are only as accurate as the number of objects an object is described relative to. In other words to describe something one must have at least a standard to compare it to. For example when Beavis and Butthead rate an early 80s video as, “This Sucks!”, what they really mean is relative to AC/DC or Metallica the video indeed, “sucks.”; however, a fan of that particular 80s group would say, “This rules”. about the video, since for him or her the video is the ultimate and anything else is described relative to it. So AC/DC or Metallica would probably “suck.” relative to their favorite videos.

As we mentioned before descriptions are only as accurate as the number of objects it is compared relative to. For this reason calling Triangle A average was more accurate then calling it large or small. This is because, in describing Triangle A as average we were comparing it relative to two triangles and not just one as in the other two cases. Similarly if Beavis and Butthead were to develop a more diverse, open and varied musical taste you would probably find them saying a bit more interesting things about the videos.

The importance of understanding relativity cannot be understated in describing any situation, phenomena, object or attribute. To name but a few examples from the natural world, weight, velocity, size, distance and objects and light. Relativity can also be taken a step further to ask what in this world is absolute, or whose existence is independent of anything else. In the SI units of measurement, every quantity except length, time and mass are considered derived. For example volume, density, force etc., can all be expressed in terms of the fundamental quantities of length, time and mass. But are length, mass and time, independent of each other?

Returning back to our discussion of the Sine function we saw that the Sine of an angle was simply the opposite side over the hypotenuse. We also now see how all other trigonometric functions can be expressed as variations of the Sine function alone, with the help of the Pythagorean theorem. Therefore before continuing let us go through a simple proof of the Pythagorean theorem, a theorem which as you already know, arises almost everywhere in Mathematics.

We begin this proof by drawing an altitude in a right triangle ABC,
The altitude with height, \( h \), divides the triangle into two other right triangles, both similar to triangle ABC.

Viewing these three triangles together:

As these triangles are all similar we can state two identities:

\[
\frac{a}{c} = \frac{c_2}{a} \quad \text{or} \quad a^2 = c \cdot c_2 \\
\frac{b}{c} = \frac{c_1}{b} \quad \text{or} \quad b^2 = c \cdot c_1
\]

Adding \( a^2 \) with \( b^2 \) gives us:

\[
\begin{align*}
a^2 + b^2 &= cc_1 + cc_2 \\
a^2 + b^2 &= c(c_1 + c_2) \\
a^2 + b^2 &= c(c) \quad [c_1 + c_2 = c] \\
a^2 + b^2 &= c^2
\end{align*}
\]

Taking the theorem a step further we can prove the following important identity relating sine with cosine. In any right triangle ABC

\[
\sin(\theta) = \frac{\text{opp}}{\text{adj}}
\]
\[ \sin(\theta) = \frac{b}{c} \]

Cos (q) would then be equal to a/c; however by the Pythagorean theorem; \( a^2 + b^2 = c^2 \), we can then re-write a as \( \sqrt{c^2 - b^2} \)

Hence:

\[ \cos(\theta) = \frac{\sqrt{c^2 - b^2}}{c} \]

Squaring both sides gives us:

\[ \cos^2(\theta) = \frac{c^2 - b^2}{c^2} \]

\[ \cos^2(\theta) = 1 - \frac{b^2}{c^2} \]

Remember that

\[ \sin(\theta) = \frac{b}{c} \]

\[ \cos^2(\theta) = 1 - \sin^2(\theta) \]

\[ \cos(\theta) = \sqrt{1 - \sin^2(\theta)} \]

This identity holds true for any right triangle. For remembrance sake it can also be written as; \( \sin^2(\theta) + \cos^2(\theta) = 1 \). This useful identity tells us that even the Cosine of an angle can be expressed in terms of the Sine of the angle, therefore the Sine function is the fundamental Trigonometric function, as all other functions can be derived from it, including the Cosine function.

Until now we have restricted our study to triangles, where the Sine of an angle is the ratio of the opposite side to the hypotenuse. We can therefore define the Sine function to be the function that outputs a unique ratio for an inputted angle. But we must ask ourselves what is an angle, and how can we express the sine of an angle mathematically without having to look up Trig. Tables? The solution is left for the next chapter that will go into detail to define the Sine function mathematically by using a circle and arclength.

**Chapter 18 - The Sine Function - Definition**

A. Remember a function is by definition only a function when the output is directly related to the input. For example the following function gives the Volume of water in a bowl as function of height:
\[ V_{ol}(h) = \frac{1}{3} \pi \cdot h^2 (15 - h) \]

Input any height and one can easily calculate the volume. However the function,

\[ \sin(\theta) = \frac{\text{opp}}{\text{hyp}} \]

is not a function as none of the variables; opposite, hypotenuse, or angle (theta) are related in any way. Give me an angle and I can not give you the Sine of it, that is without looking it up in some table.

To begin our study of the Sine function, we need to first find a way of defining it as a mathematical function, such that the Sine of an angle is a function of the angle inputted. It therefore behooves us to define what exactly is an angle. Till now we have assumed an understanding of angles and we looked at Triangles to show what they mean. Before defining what an angle is, let us now take an important look at circles to see how angles can be defined by them.

From the Pythagorean theorem we showed that in any right triangle, \(a^2 + b^2 = c^2\). The equation:

\[ x^2 + y^2 = r^2 \]

Is therefore the set of all points a distance \(r\) from the origin, from the distance formula which is based on the Pythagorean theorem. The important point to realize here is that the co-ordinates of each point on the circle gives us the dimension of a unique right-triangle with base, \(x\), height, \(y\) and hypotenuse \(r\). Each point satisfies the Pythagorean theorem for right triangles, \(a^2 + b^2 = c^2\); though the radius or hypotenuse remains fixed, the \(x\) and \(y\) can take on infinite values.

To visualize these concepts, the Graph of the circle \(x^2 + y^2 = r^2\)
Let us now look at how to define an angle in terms of the circle. An angle in all simplicity is the measurement of rotation between two intersecting lines. This rotation can be expressed in terms of the arc length of the circle. If you follow the path of a point at (3,0) to (0,3) along the arc of the circle then you will notice that the radius drawn to each point on the path will increase at a steady rate.

You can notice two points here. First the circumference of this circle is given by the formula:

\[ C = 2 \cdot \pi \cdot r \]

This tells us the total arc length of this circle is just \( 2 \cdot \pi \cdot r \), where \( r \) is the radius. Now if we define there to
be 360 degrees in a circle, what this means is that a line that rotates a full circle to return to its original position will have covered 360 degrees (a completely arbitrary number). Since degrees are a measurement of an angle, or the amount a line or \( r \) in the graph has rotated then we must figure out some method of relating the arc length, a measurement of rotation to degrees.

If we define there to be 360 degrees in one full rotation the ratio of some number of degrees \( \theta \) to 360 degrees must equal the ratio of the arc length covered to the total arc length of the circle.

\[
\frac{\text{arclength}}{2 \cdot \pi \cdot r} = \frac{\text{Degrees}(\theta)}{360}
\]

Multiplying terms out to solve for degrees gives us:

\[
\text{Degrees}(\theta) = 360 \cdot \frac{\text{arclength}}{2 \cdot \pi \cdot r}
\]

There is still one thing left, this definition of degrees is entirely dependent on the radius of the circle. If you recall from our discussion of the relativity of size in mathematics as being meaningless without reference to something else, then theoretically 30 degrees should be independent of the size of the circle it is measured in. To account for this we introduce an important new concept, the radian. The radian is a relativistic measurement for arc length that gives us the arc length measurement in terms of the radius of the circle. For example in a circle of radius 2, the circumference is \( 4 \pi \). An arc length of 2 radians is then just 4 units long or twice the radius. Since the circumference of the circle is directly related to its radius then an arc length in radians will always have the same relative size to its circle, regardless of the size of the circle. This probably sounds more confusing than it really is, but this is all because we are looking for a way to define degrees in terms of arc length. If we replace arc length with radians*\( r \), where \( r \) represents the constant radius and radians is any number which could be a fraction we get:

\[
\text{Degrees}(\theta) = 360 \cdot \frac{\text{Radians} \cdot \theta}{2 \cdot \pi \cdot r}
\]

This important equation now tells us how to define an angle in terms of the radian measurement of an arc length, measured in radians. We can re-write it as:

\[
\text{Radians}(\theta) = \frac{2 \cdot \pi}{360} (\theta)
\]

The important point to realize throughout this discussion of what an angle is, is that angles can only be first defined in terms of arc length rotations between two lines measured in radians. Having established radians, only then can we also use degrees, which is just another unit of measurement of angles entirely based on radians. Instead of calling an angle .86 radians we can say 60 degrees:

\[
\theta(r) = 360 \cdot \frac{r}{2 \cdot \pi} \quad \text{where} \quad 0 \leq r \leq 2\pi \quad \text{and} \quad 0 \leq \theta \leq 360
\]

Degrees are directly related to radians, and are more often used when there is no reference circle to define an angle easily.

Let us now return to our discussion of circles. Having established just what an angle refers to conceptually we now need to define an angle mathematically. We shall select for our study a circle of radius one, which is often called the unit-circle, for reasons soon to be seen. If we draw a small triangle in the unit-circle we can see that:
In such a triangle

\[
\begin{align*}
\cos(\theta) &= \frac{x}{1} = x \\
\sin(\theta) &= \frac{y}{1} = y \\
\end{align*}
\]

What this means is that for any point \((x, \sqrt{1-x^2})\) on the unit circle, the sine of the arc length measured from \((0,0)\) to \((x, y)\) will be the y value of that particular arc length. For example:

\[
\sin\left(\frac{\pi}{2}\right) = \frac{\sqrt{2}}{2}
\]

In terms of triangles and degrees, this says that the ratio of the opposite side to the hypotenuse of a 45 degree angle is just \(\frac{\sqrt{2}}{2}\). It is extremely important to forget about triangles and degrees for the moment but just concentrate on circles and arc lengths which are both mathematically defined shapes. In terms of the unit circle whose the angle measurement 45 degrees is expressed in terms of the arc length of a circle whose radius is \(\frac{\pi}{2}\) then refers to an arc length of 1.56 radii.
We are now ready to define the Sine function as that function that outputs the y-value for an inputted arc length of a unit circle. The cosine functions gives us the x-value for that same arc length. It is here where the difficulty arises. We need to find a way of expressing the arc length of a circle a unit circle in terms of x and y. By being able to calculate the arc-length in terms of x and y values, the Sines and Cosines will just be the x and y values of the point until which we are calculating the arc length. Therefore the Sine function is really an arc length function as we shall soon see.

To begin let us recall our formula for calculating the length of curves of graphs of equations. It was defined as the following differential that gives the length of a hypotenuse used to approximate the graph over a small interval of x.

$$\Delta Arclength = \sqrt{1+{(f'(x))^2}} \cdot \Delta x$$

Integrating it to find the total arc length.

$$Arclength = \int \sqrt{1+{(f'(x))^2}} \cdot \Delta x$$

For example the length from 0 to 1 of the following graph of \(y=5x\) is:
The length of the graph from 0 to 1 therefore has length 5.22.

Relating this to our study of the circle, the arc length can be expressed as follows:

Equation of Unit Circle is \( x^2 + y^2 = 1 \),
\[ y = \sqrt{1 - x^2} \]

The derivative of this function for a circle is:
The arc length from zero to \( a \), where \( a \leq 1 \), is:

\[
f(x) = \sqrt{1 - x^2} \text{ or } \left(1 - x^2\right)^{1/2}
\]

\[
f'(x) = \frac{1}{2(1 - x^2)^{3/2}} \cdot -2x
\]

\[
f''(x) = \frac{-x}{\sqrt{1 - x^2}}
\]

The arc length from zero to \( a \), where \( a \leq 1 \), is:

\[
\text{Arclength} = \int_{0}^{a} \sqrt{1 + (f'(x))^2} \, dx
\]

In the circle this translates to:

\[
f(x) = \sqrt{1 - x^2} \text{ or } \left(1 - x^2\right)^{1/2}
\]

\[
f'(x) = \frac{1}{2(1 - x^2)^{3/2}} \cdot -2x
\]

\[
f''(x) = \frac{-x}{\sqrt{1 - x^2}}
\]

There are two important points to understand here. First, for any \( 0 \leq x \leq 1 \), the output will be an arc length measured in radians since we are dealing with the unit circle. For example, if I were to evaluate the integral from \( x=0 \) to \( x=1 \), I should get \( \frac{\pi}{2} \) or 1.57 radii. The second point to keep in mind is that \( x^2 + y^2 = 1 \) is the same as \( y = \sqrt{1 - x^2} \) and \( \frac{\Delta y}{\Delta x} \) is therefore \( \frac{-2}{2\sqrt{1 - x^2}} = \frac{-x}{\sqrt{1 - x^2}} \).

Returning to our integral we have:

\[
\text{arclength} = \int_{0}^{a} \sqrt{1 + \left(\frac{-x}{\sqrt{1 - x^2}}\right)^2} \, \Delta x
\]

\[
\text{arclength} = \int_{0}^{a} \sqrt{1 + \left(\frac{-x}{\sqrt{1 - x^2}}\right)^2} \, \Delta x
\]
Simplifying the integral:

\[
\text{arclength} = \int \sqrt{1 + \left( \frac{x^2}{1 - x^2} \right)} \cdot \Delta x
\]

The expression in the radical, \(1 + \frac{x^2}{1 - x^2}\), can be further reduced to \(\frac{1}{1 - x^2}\) by adding the two fractions:

\[
\frac{1}{1 - x^2} + \frac{x^2}{1 - x^2} = \frac{1 - x^2 + x^2}{1 - x^2} = \frac{1}{1 - x^2}
\]

This now gives us:

\[
\text{arclength} = \int \frac{1}{\sqrt{1 - x^2}} \cdot \Delta x
\]

The question now arises, how do we interpret this or relate it to the Sine function. To answer this let us study the graph of the circle once more. Since we commonly measure arc lengths in radians or degrees beginning at (1,0) it will require a moments thought to realize that now we are measuring angles from (0,1).
the Sine function as follows.

\[
\text{arclength} = \int \frac{1}{\sqrt{1 - x^2}} \cdot \Delta x
\]

\[
\text{Sine}(\text{arclength}) = x
\]

\[
\text{Sine} \left( \int \frac{1}{\sqrt{1 - x^2}} \cdot \Delta x \right) = x
\]

Since \( x \) is our output and not our input here, we need to find a better way of defining the Sine function. If you recall from our study of inverse functions, we saw that the inverse function always yields \( x \) when inputted in the original function since a function and its inverse were essentially the same function except that the \( x \) and \( y \) were switched around, and the two functions were then graphed in the same \( x-y \) plane. For example if we had \( y = x^2 \) then its inverse was found by replacing \( y \) with \( x \) to get \( x = y^2 \) or \( y = \sqrt{x} \). Since any point \((x, y)\) on the graph of \( y = x^2 \) has a corresponding point on \((y, x)\) on the graph of its inverse; \( y = \sqrt{x} \) then inputting \( y \) values from \( f^{-1}(x) \) in \( f(x) \) would output \( x \) values. For example:

\[
f(x) = x^2
\]

\[
f^{-1}(x) = \sqrt{x}
\]

\[
f(f^{-1}(x)) = \left(\sqrt{x}\right)^2 = x
\]

This may sound more complicated then it really is but we can now conclude that since the Sine of the arc length integral outputs \( x \), then that integral is the inverse of the Sine function:

\[
\text{Sine} \left( \int_0^x \frac{1}{\sqrt{1 - x^2}} \cdot \Delta x \right) = x
\]

\[
\text{Sine}^{-1}(x) = \int_0^x \frac{1}{\sqrt{1 - x^2}} \cdot \Delta x
\]

The inverse sine function is commonly called the arcsine function which is analogous to asking what angle or arc length has sin \( x \)?

\[
\text{arcsin}^{-1}(x) = \int_0^x \frac{1}{\sqrt{1 - x^2}} \cdot \Delta x
\]

\[
\sin^{-1}(x) = \int_0^x \frac{1}{\sqrt{1 - x^2}} \cdot \Delta x
\]

At this point is important to stop thinking about angles, degrees, and triangles. What is essential now are circles and arc lengths measured in radians. The Sine function, mathematically speaking is then just the function gives the corresponding \( x \) value of an arc length measured from to 0 to \( x \) on the circle. The inverse sine of a number \( x \) is then just the arc length from 0 to \( x \) on the graph of a circle. This is all it is, nothing more, nothing less. The inverse sine function is therefore our most important function since it serves as the base for calculating arc lengths for any \( 0 \leq x \leq 1 \).
This concludes our definition of the Sine function as mathematical function to express the arc length of a unit circle, or any circle. You can already begin to see the problems we will encounter in integrating it since the arc length approach infinity towards \( x=1 \).

**Chapter 19 - Differentiating and Integrating the Sine function**

In the previous chapter we underwent a long and thorough analysis of just defining the Sine function mathematically. This was done with by calculating the arc length of a circle, the inverse sine of \( x \) being the arc length from zero to \( x \). The goal of this chapter is to differentiate and integrate this function and as we shall soon see, the sine function will be unlike any other function studied so far.

From the definition of the inverse sine or arcsine function we have:

\[
\sin^{-1}(x) = \frac{x}{0 \sqrt{1-x^2}} \cdot \Delta x
\]

\[
\pi = 2 \cdot \int_{0}^{1} \frac{1}{\sqrt{1-x^2}} \cdot \Delta x
\]

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From the definition of the inverse sine or arcsine function we have:

\[
\sin^{-1}(x) = \int_{0}^{1} \frac{1}{\sqrt{1-x^2}} \cdot \Delta x
\]

Once again if you recall from the Chapter on inverse we learned that a function and its inverse are essentially
the same function. We can therefore easily find an expression for \( \sin(x) \) based on the definition for \( \sin^{-1}(x) \).

The only problem here is that we have to integrate before we can find a way of expressing \( \sin(x) \). We do however know how to calculate the derivative of \( \sin^{-1}(x) \). From the definition of the integral it is:

\[
\sin^{-1}(x) = \int_{0}^{x} \frac{1}{\sqrt{1 - t^2}} \cdot \Delta t
\]

\[
\frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{1 - x^2}} \text{ (Derivative of } \sin^{-1}(x))
\]

By following the algorithm presented in the chapter on inverses for finding the derivative of a function’s inverse we can find the derivative of the Sine function without having found a way to define it yet.

We first replace any \( x \) with its \( y \) equivalent so as to be able to find the derivative with any given \( y \)-value.

\[
\frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{1 - x^2}}
\]

\[
y = \sqrt{1 - x^2}
\]

\[
\frac{\Delta y}{\Delta x} = \frac{1}{y}
\]

From the graph of the circle this tells us that the derivative of the inverse sine function is just 1 over the \( y \) value of that particular arc length.

From the graph you can see that this value corresponds to the Cosine of the arc length or angle by the definition of the Cosine Function. Therefore:

\[
\frac{\Delta y}{\Delta x} = \frac{1}{y}
\]

\[
y = \cos(\theta)
\]

\[
\frac{\Delta y}{\Delta x} = \frac{1}{\cos(\theta)}
\]

There is one important difference to keep in mind. The \( \Delta y \) in the left side refers to the derivative of the
inverse Sine function or arc length function of x. Therefore \( \Delta y \) means a small change in the arc length. On the other hand the y in the \( y = \cos(\theta) \) is just referring to the point until which this arc length is being measured in terms of the unit circle. We can now write:

\[
\sin^{-1}(x) = \frac{\Delta y}{\Delta x} = \frac{\Delta \text{Arclength}}{\Delta x} = \frac{1}{\cos(\text{Arclength})}
\]

\[\Delta \theta = \frac{1}{\cos(\theta)}\]

Remember arc length and \( \theta \) are the same thing from our definition of the angle. In the graph of the inverse sine function, x is our independent variable and \( \theta \) or arc length is our dependent variable. Since the Sine function is the inverse of the inverse Sine function, then \( \theta \) is our independent variable and x is our dependent variable. This means that the derivative of the Sine function is given by:

\[
\frac{\Delta x}{\Delta \text{Arclength}} = \frac{\Delta x}{\Delta \theta}
\]

This tells us that the derivative of the Sine function is just the Cosine of that particular angle or arc length. The Sine function inputs arc lengths and outputs the corresponding x-value of that arc length. Therefore its derivative with respect to arc length is just the Cosine of that arc length.

To find the derivative of the Cosine function first recall that

\[
\cos(\theta) = \sqrt{1 - \sin^2(\theta)}
\]

\[
\frac{\Delta y}{\Delta \theta} = \frac{1}{2} \cdot \frac{1}{\sqrt{1 - \sin^2(\theta)}} \cdot -2\sin(\theta) \cdot \cos(\theta) = \sqrt{1 - \sin^2(\theta)}
\]

\[
\frac{\Delta y}{\Delta \theta} = -\sin(\theta)
\]

This tells us that the derivative of the Cosine function with respect to arc length is just the negative Sine of that arc length. To summarize:

\[
f'(\theta) = \sin(\theta)
\]

\[
f''(\theta) = \cos(\theta)
\]

\[
f'''(\theta) = -\sin(\theta)
\]
We have now found the derivative of the Sine function in terms of the Sine and Cosine of the arc length, $\theta$. There still remains the stumbling block, what is $\theta$ or arc length? Though we have shown rather crudely what is the derivative of the Sine and Cosine function are, we still have not found a way of defining $Sin^{-1}(\theta)$. The only obstacle was our definition for the inverse function:

$$Sin^{-1}(x) = \int_{0}^{x} \frac{1}{\sqrt{1-x^2}} \cdot \Delta x$$

There is absolutely no way we can evaluate this integral. You can understand this in two ways. First the derivative approaches infinity as x goes to 1 and second from the graph of the circle, arc length is continuous never ending length. It goes round and round the circle forever. Though we have defined a beginning there is no end. It is this complex nature of the arc length of a circle that tells us that there is no way to solve its using real numbers.

It is what we call an irrational integral and can only be solved with complex or imaginary numbers, $i = \sqrt{-1}$. The best way of thinking of what imaginary numbers are is that they are numbers that basically tell us there is no solution to the equation. There is nothing magical or transcending about them. If we ever see them, they only alert is that the equation has no solutions.

Using complex numbers the integration goes as follows:

First factor out -1 in the denominator:

$$Sin^{-1}(x) = \int_{0}^{x} \frac{1}{\sqrt{-1(-1 + x^2)}} \cdot \Delta x$$

$$= \int_{0}^{x} \frac{1}{\sqrt{-1} \cdot \sqrt{x^2 + 1}} \cdot \Delta x$$

Factoring out a $i$ gives us:

$$= \frac{1}{i} \int_{0}^{x} \frac{1}{\sqrt{x^2 + 1}} \cdot \Delta x$$

$$= \frac{1}{i} \left[ \ln \left( x + \sqrt{x^2 + 1} \right) \right]$$

The derivative of $\ln \left( x + \sqrt{x^2 + 1} \right)$ is:
We have shown how \( y = \ln \left( x + \sqrt{x^2 - 1} \right) \). Now let us find an expression for \( \sin^{-1}(x) \). Since \( \sin^{-1}(x) \) is the inverse of \( \sin(x) \), we only need to interchange \( y \) and \( x \) to solve for \( y \), where \( y = \text{arc length} \):

\[
\Delta y = \frac{1}{x + \sqrt{x^2 - 1}} \left( 1 + \left( \frac{x}{\sqrt{x^2 - 1}} \right) \right)
\]

\[
\Delta y = \frac{1}{x + \sqrt{x^2 - 1}} \left( \frac{\sqrt{x^2 - 1} + x}{\sqrt{x^2 - 1}} \right) = \frac{\sqrt{x^2 - 1} + x}{\sqrt{x^2 - 1} + x} \cdot \frac{1}{\sqrt{x^2 - 1}}
\]

\[
\therefore \frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{x^2 - 1}}
\]

Factoring out a \( i \) or \( -1 \) from \( \sqrt{y^2 - 1} \) gives us:

\[
e^{ix} = y + i \cdot \sqrt{1 - y^2}
\]

where \( y = \cos(x) \) and \( \sqrt{1 - y^2} = \sqrt{1 - \cos^2(x)} = \sin(x) \). This now reduces to the famous Euler’s equation:

\[
e^{ix} = \cos(x) + i \cdot \sin(x)
\]

You might object to our replacing \( y \) with \( \sin^{-1}(x) \). Do not forget that when \( \sin^{-1}(x) \) referred to arc length or \( \sin^{-1}(x) \) and i interchanging \( x \) with \( y \), \( y \) then refers to \( \sin(x) \) since that is what we were initially calculating.

Euler’s equation probably makes little sense to you. All it really is is another way of expressing the relationship between the complex logarithm and circular functions (Stillwell 221). The sine and cosine functions are therefore solutions to such irrational integrals.

There is still one way of defining \( \sin^{-1}(x) \) which is through an infinite series of terms based on the binomial expansion theorem, which defines an infinite series for sums raised to negative and fractional exponents:

\[
(a + b)^n = a^n + \frac{n(n-1)}{1!} a^{n-1} b^1 + \frac{n(n-1)(n-2)}{2!} a^{n-2} b^2 + \cdots + \frac{n(n-1)(n-2)\cdots(n-(n-1))}{n!} a^{n-n} b^n
\]

or more simply:
Though we will study infinite series in more detail in the following chapter, it would do no harm to begin the study with our inverse Sine dilemma.

As the expansion for $(1 + x)^n$ is valid for all values of n, then we are able to write the integral for $\sin^{-1}(x)$ in a different way:

$$\sin^{-1}(x) = \int_0^x \frac{1}{\sqrt{1 - x^2}} \cdot dx$$

Which now becomes:

$$\sin^{-1}(x) = \int_0^x \left[1 + \left(-x^2\right)^{\frac{1}{2}}\right] \cdot dx$$

The binomial expansion is:

$$\left(1 + \left(-x^2\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} = 1 + \left(\frac{1}{2}\right)(-x^2) + \left(-\frac{1}{2}\right)(-\frac{3}{2})(-x^2)^2 + \ldots$$

$$\left(1 + \left(-x^2\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} = 1 + \left(-\frac{x^2}{2}\right) + \left(-\frac{x^4}{8}\right) + \left(-\frac{5x^6}{18}\right) + \ldots$$

This is clearly an infinite series with never ending terms since $x^n$ will never equal $\frac{1}{x}$, Integrating the series will give us the series for $\sin^{-1}(x)$:

$$\sin^{-1}(x) = \int_0^x \left[1 + \left(-x^2\right)^{\frac{1}{2}}\right] \cdot dx = \int_0^x (1 \cdot dx + \left(-\frac{x^2}{2}\right) \cdot dx + \left(-\frac{x^4}{8}\right) \cdot dx + \left(-\frac{5x^6}{18}\right) \cdot dx + \ldots + \left(-\frac{1 \cdot 3 \cdot 5 \ldots (2n - 3) \cdot x^{2n-2}}{2 \cdot 4 \cdot 6 \ldots (2n - 2)}\right) \cdot dx$$

The series is valid for $-1 < x < 1$. For $|x| > 1$, each term in the series will get larger, causing the series to diverge. We will talk more about how this series works in the next chapter.

**Summary**

In these three lengthy chapters we undertook an in depth study of defining the Sine function. Beginning with the simplest definitions we moved on to defining its inverse function as an arc length:

$$\sin^{-1}(x) = \int_0^x \frac{1}{\sqrt{1 - x^2}} \cdot dx$$

The $\sin^{-1}(x)$ is in reality the arc length of a circle as a function of x or:
We showed how angles can only be measured in radians, a relative measurement that relates arc length to the rotation of two lines and the angle they define. Thus:

\[ \text{Arclength}(x) = \sin^{-1}(x) \]

This relates degrees, measured in radians, as a function of \( x \).

The integral being “un-integrable” due to the unique properties of the circle, could only be solved with imaginary numbers. We then went on to prove the complex relationship between the exponential function and the Sine function, or \( e, \pi, i \).

Last but not least we were shown, rather simply, how the inverse sine function can be expressed as an infinite series of terms based on the binomial theorem for negative fractional exponents. Our study of the Sine function will resume in the next chapter when we study the function in yet another way, actually the same way but by using some new theorems from Calculus.

**Chapter 20 - Applications of the Sine Function - Oscillatory Motion**

Having gone through a long and detailed study of the Sine function the time arrives for us to study its applications. There are many applications of the Sine function, however, the most common are relate the dynamics of vibrating or oscillating systems. By oscillating system we mean any object whose motion can be characterized as a continuous back and forth never ending motion. Such motion occurs when the resistance to motion of a body in motion is directly proportional to the distance covered and acts in a direction that is opposite to the direction of movement.

The typical and simplest example of this is just a simple spring with a body or block of mass \( m \) attached to it.

If we assume that the block with mass \( m \) rests on a frictionless surface then the block is free to oscillate with any horizontal force. According to Hooke’s law the force required to stretch a spring is directly proportional to the length of the spring. Each spring therefore has a constant, known as the spring constant \( k \) which tells us the force required to stretch the spring a given distance.

For example if the spring constant, \( k \), of a rather stiff spring such as those used in cars as opposed to a slinky, were 1800 newtons per meter or \( k = 1800 \text{ N/m} \) then to stretch the spring one meter would require a force of 1800 Newtons and to further stretch it three meters would require a force three times as great or 5400 N. This is all in accord with Hooke’s Law, which states that the more you stretch it the greater the force required.

Consider now the block above of mass \( m \) at equilibrium position. By equilibrium position we mean the object is at rest with no unbalanced force acting on it. This tells us that the spring is being neither compressed nor stretched but is just lying there attached to the block. If we stretch the block a distance \( x \) from its equilibrium position and let go of it, then it will oscillate back and forth. Our goal is to mathematically describe this motion.
The amplitude of oscillation is 2x and the maximum force exerted by the spring on the block is kx. Keep in mind however that the force is zero at the equilibrium position, since the spring is un-stretched there and at this point the block's entire energy is kinetic while at the endpoints of its oscillation it possesses entirely potential energy stored in the spring and its velocity is zero.

Since the spring exerts a force kx when stretched, the work done stretch a spring a distance x is given by:

\[
W = \text{Force} \times \text{Distance} \\
W = F(x) \cdot d \\
\Delta W = k \cdot d \cdot \Delta d \\
W = \int k \cdot d \cdot \Delta d \\
\int W = \frac{k}{2} d^2 \quad (d = x) \\
W = \frac{k}{2} x^2
\]

The work required to stretch a spring a distance x is equal to \( \frac{k}{2} x^2 \). When any mass m is stretched a distance x then let go of it will be pulled back with a force that varies with the distance stretched.

The total energy of the system varies between kinetic and potential energy depending on the mass's position. The kinetic energy of the block in motion is always equal to \( \frac{1}{2} m v^2 \), where v is its velocity at a particular point. At that point x the potential energy will be \( \frac{1}{2} kx^2 \) or the energy used to stretch the spring to that position. The sum of the potential and kinetic energy of the block at any point is a constant that never changes. It corresponds to the net energy of the oscillating system:

\[\frac{1}{2} kx^2 + \frac{1}{2} m v^2 = B\]

At x=0, the velocity is a maximum and when v=0, x is a maximum and the block is at its endpoint.

\[\frac{\text{distance}}{\text{time}} = \frac{\Delta x}{\Delta t}\]. Let us first solve the equation for v:

\[\frac{1}{2} kx^2 + \frac{1}{2} m v^2 = B\]
Multiplying both sides by 2 gives:

\[ kx^2 + mv^2 = 2E \]

\[ mv^2 = (2E - kx^2) \]

\[ v^2 = \frac{1}{m} (2E - kx^2) \]

\[ v = \sqrt{\frac{1}{m} (2E - kx^2)} \]

\[ \frac{\Delta x}{\Delta t} = \frac{1}{m} (2E - kx^2) \]

Now substituting velocity for \( \frac{\Delta x}{\Delta t} \) and separating variable yields:

\[ \frac{1}{\Delta x} \Delta x = \Delta t \]

Integrating both sides:

\[ \int \frac{1}{\Delta x} \Delta x = \int \Delta t \]

\[ \int \frac{1}{\sqrt{m} \sqrt{2E - kx^2}} \Delta x = t \]

or

\[ t = \sqrt{m} \int \frac{1}{\sqrt{2E - kx^2}} \Delta x \]

Recall that \( E \) stood for the total energy of the system. At its endpoints the velocity of the block is zero. The total energy is then equal to \( \frac{1}{2} \frac{k\chi_{\text{max}}^2}{2} \) where \( \chi_{\text{max}} \) equals half the amplitude of oscillation.

\[ E = \frac{1}{2} k \left( \frac{A}{2} \right)^2 \]

\[ E = \frac{1}{8} kA^2 \]

Substituting back into our integral gives us:
\[ t = \sqrt{\frac{1}{(2E-kx^2)}} \cdot \Delta x \]

\[ t = \sqrt{\frac{1}{\left(2\left( \frac{1}{8}kA^2 \right) - kx^2 \right)}} \cdot \Delta x \]

\[ t = \sqrt{\frac{1}{\left( \frac{1}{4}kA^2 - kx^2 \right)}} \cdot \Delta x \]

We can factor out a \( k \) from the denominator to get:

\[ t = \sqrt{\frac{1}{k \left( \frac{1}{4}(A^2 - 4x^2) \right)}} \cdot \Delta x \]

\[ t = \sqrt{\frac{1}{\frac{k}{4}(A^2 - 4x^2)}} \cdot \Delta x \]

We can take out the constant \( \sqrt{\frac{k}{4}} \) from the integral to get:

\[ t = \frac{4}{k} \sqrt{\frac{m}{(A^2 - 4x^2)}} \cdot \Delta x \]

\[ t = 2 \frac{m}{k} \sqrt{\frac{1}{(A^2 - 4x^2)}} \cdot \Delta x \]

We can now factor out \( A^2 \) from \( (A^2 - 4x^2) \) and then remove the constant \( A \) from the integral to get:

\[ t = 2 \frac{m}{k} \sqrt{\frac{1}{A^2 \left( 1 - \frac{4}{A^2}x^2 \right)}} \cdot \Delta x \]

\[ t = 2 \frac{m}{k} \sqrt{\frac{1}{A^2 \left( 1 - \frac{4}{A^2}x^2 \right)}} \cdot \Delta x \]

\[ t = \frac{2}{A} \sqrt{\frac{m}{\left( 1 - \frac{4}{A^2}x^2 \right)}} \cdot \Delta x \]
This integral now has the familiar solution that we have studied:

\[ \int \frac{1}{\sqrt{1-cx^2}} = \frac{1}{c} \sin^{-1}(\sqrt{c}x) \]

Since the derivative of \( \sin^{-1}(u) = \frac{1}{\sqrt{1-u^2}} \frac{\Delta u}{\Delta x} \)

\[ a = \frac{4}{A^2} \quad \text{and} \quad \sqrt{a} = \frac{2}{A} \]

\[ t = \frac{2m}{\sqrt{k}} \int \frac{1}{\sqrt{1 - \frac{4}{A^2} x^2}} \cdot \Delta x \]

has the solution:

\[ \frac{A}{2} \cdot \frac{2m}{\sqrt{k}} \sin^{-1}\left(\frac{2x}{A}\right) = t \]

\[ \frac{m}{k} \sin^{-1}\left(\frac{2x}{A}\right) = t \]

Now that we have solved it, how do we interpret it or make sense of this solution? Basically this solution tells us that the time required for the mass to a distance \( x \) from the equilibrium position. We can therefore easily calculate the period of the oscillation, the time required for the mass to complete one cycle. The period would be four times the time required for the mass to move from \( x=0 \) to \( x= \frac{A}{2} \). Substituting this into our solution gives:

\[ \text{Period} = 4 \left( \frac{m}{k} \sin^{-1}\left(\frac{2}{A} \cdot \frac{A}{2}\right) \right) \]

\[ \text{Period} = 4 \left( \frac{m}{k} \sin^{-1}(1) \right) \]

When \( x = 1 \), the arc length is \( \frac{\pi}{2} \)

\[ \text{Period} = 4 \left( \frac{m}{k} \cdot \frac{\pi}{2} \right) \]

\[ \text{Period} = 2\pi \ \frac{m}{k} \]

This answer tells us that the period of an oscillating mass on a spring is independent of the amplitude of the oscillation. For this reason springs are used in watches. Despite air-resistance and friction between the spring and other parts that would cause the Amplitude to gradually decrease over time, the period of vibration would remain unchanged.

Just as we are able to find time as a function of \( x \), we can find \( x \) as a function of time using the Sine function
or the inverse Sine Function. We saw that:

\[
t = \sqrt{\frac{m}{k}} \sin^{-1}\left(\frac{2x}{A}\right)
\]

Multiplying both sides by \(\sqrt{\frac{k}{m}}\) gives us:

\[
\sqrt{\frac{k}{m}} t = \sin^{-1}\left(\frac{2x}{A}\right)
\]

Taking the Sine of both sides just switching around the dependent and independent variable:

\[
\sin\left(\sqrt{\frac{k}{m}} t\right) = \frac{2x}{A}
\]

\[
x = \frac{A}{2} \sin\left(\sqrt{\frac{k}{m}} t\right)
\]

(Many physics books replace \(\sqrt{\frac{k}{m}}\) with \(\omega\), where \(\omega = 2\pi f\) and \(f\), frequency = \(1/t\), \(t\), period = \(\frac{2\pi}{\omega}\). The reason being \(\omega\) stands for radians/sec and hence keeps the units in order and simplifies calculations.)

Now that we have the position \(x\) as a function of time we can differentiate the function one to find the velocity and twice to find the acceleration.

\[
distance = x = \frac{A}{2} \sin\left(\sqrt{\frac{k}{m}} t\right)
\]

\[
\frac{\Delta x}{\Delta t} = \text{velocity} = \frac{A}{2} \sqrt{\frac{k}{m}} \cos\left(\sqrt{\frac{k}{m}} t\right)
\]

\[
\nu''(t) = \text{acceleration} = \frac{A}{2} \frac{k}{m} \sin\left(\sqrt{\frac{k}{m}} t\right)
\]

This expression \(x = \frac{A}{2} \sin\left(\sqrt{\frac{k}{m}} t\right)\) in \(\frac{\Delta x}{\Delta t}\) with its original reference, \(x\) or position since \(x = \frac{A}{2} \sin\left(\sqrt{\frac{k}{m}} t\right)\). This now gives us:

\[
\text{acceleration} = \frac{k}{m} \frac{\Delta x}{\Delta t}
\]

Since Force is by definition mass* acceleration, if we multiply the function for acceleration by \(m\) we get:
Force = mass × acceleration = $-\frac{k}{\eta} \cdot \dot{x} \cdot \ddot{x}$

$Force = -kx$

This is what we originally started with, Hooke’s Law. The force in a spring is proportional to the distance it is stretched and acts in the opposite direction of the motion to oppose the motion.

Before ending you might be wondering what happened to the constants? I decided not to put them in as to avoid any unnecessary variables or numbers that would impede your understanding of the important concepts being studied. The solutions with the constants included should here forth read:

\[
\begin{align*}
x(t) &= \frac{A}{2} \sin(\omega \cdot t + \delta) \\
\dot{t}(x) &= \sqrt{\frac{m}{k}} \sin^{-1}\left(\frac{2}{A} x + x_i\right)
\end{align*}
\]

Summary

In this chapter we saw how in a spring the force required to stretch the spring is directly proportional to the length stretched. Recalling that force is defined as mass* acceleration or $F = ma$, where acceleration is the second derivative of the position function of an object in motion:

\[
\begin{align*}
Force &= -kx \\
m \cdot x''(t) &= -kx \\
x(t) &= -\frac{k}{m} x
\end{align*}
\]

This can be rewritten as

\[
x''(t) = -kx
\]

The only function whose second derivative is proportional to negative of the original function is the Sine function, since:

\[
\begin{align*}
y &= \sin(x) \\
y' &= \cos(x) \\
y'' &= -\sin(x) \\
y'' &= -y
\end{align*}
\]

Therefore this chapter was in essence devoted to the understanding of how the Sine function and its derivative apply to applications where the second derivative is directly proportional to the negative of the original function. This is why the Sine function is restricted to Oscillatory motion, since a force -kx, will always act so as to oppose the motion of the body in any direction. From the Conservation of Energy the body will then rebound back in the opposite direction once whatever force acting on it brings its velocity to zero because all the energy will now have been transferred from the mass to the system. In such a way this back and forth process goes on forever until interrupted by other forces such as friction.