Understanding Calculus

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Contents

Part 2

Preface

The purpose of this book is to present Mathematics as the Science of Pure Reasoning and not as the Art of Manipulation. The plethora of textbooks available today, are riddled with complex and abstract jargon. Their goal is not to impose a lasting understanding of and appreciation for Calculus on the students, but rather to present Mathematics as something incomprehensible to any human being.

Theorems are never explained as to how they came into being. They are simply stated and are meant to be taken for granted. Where the book feels a proof is needed, a highly complex proof is presented with over six different variables floating about. No explanations accompany any such proof, as the textbook deems them as self-explanatory, which they clearly are not. The student has a difficult time in figuring out just what the proof says, much less what it means. Obviously the use of such mathematical jargon serves only to confuse the student and has no place in an elementary Calculus book intended for high-school and college students.

Furthermore, these empty theorems that spring from every page are never given any example of practical applications. Not only does the student not understand or know what he is learning is coming has come from but neither does he know where it is going. The textbook's idea of exercises is nothing more than sheer number-crunching and manipulation of variables. The entire underlying principle of order and beauty upon which Calculus is based, is neglected. The problems never call upon the student's ability to think logically. Rather, they require no more than time and persistence, to allow the trial and error method to succeed.

It is clear that the above scenario leaves the student with no more than two choices. One, he or she can burden his or her mind out in trying to make sense of this mysterious science, where theorems appear out of nowhere and seem to have no connection with previously learned material. Or second, he or she can choose to memorize. Without a doubt, the majority of students will opt for the later. Not only is it less time-consuming, but it also yields the correct answer.

One may argue that it is not important for students to understand what they are learning. The goal is for the student to be able to quickly learn formulas and theorems and apply them. Unfortunately it is impossible for a student to apply Calculus in the real-world, when he or she does not understand what exactly the theorems mean.

The over-emphasis on the calculator and foremostly the computer is yet another point of confusion for the student. It is beyond me how graphing utterly insane functions such as $f(x) = ln\ (sin\ (x)),^{tan\ x}$ strengthens a students conceptual understanding of Calculus. The computer is only a time-saving machine whose usefulness depends on the knowledge of the user. We can not expect it to provide that knowledge to us as it is we who have to tell it what to do. I do admit the computer is a remarkable machine, yet it is this fascination that gives students a false sense of what they are doing. The confidence gained from all the correct answers leads to an inseparable dependence on the machine where the student is absolutely helpless with out it. Similar sentiments can be said about the Calculator.

It is all these faults that I set out to correct in writing my book. The book I have chosen to write is a reaction against this empty science of Mathematics that the textbooks are teaching. My book is intended to give the student a real for and understanding of what Calculus is truly about. Only by explaining where something has come from will I be able to show where it is going. It does not take more intelligence than that of a parrot to be able to go through a list of theorems and equations; but only when one understands their origins can one correctly and confidently apply them in the real world.

I assume absolutely nothing and neither do I take anything for granted. Each chapter has a definite beginning, followed by logical, elegant and clear proofs, concluded with a brief summary that ties everything together. My only reference has been reason and if you could find just three lines that remotely resemble the lines from any other book, I would feel greatly distressed.

Throughout the textbook I constantly refer to science and engineering. The purpose of this to show how the scientific method applies to all disciplines and to understand that mathematics is an expression of one’s observations and hypothesis. I do not overdose the student with series of real-world applications where Calculus is applied. Doing so would only succeed in showing that calculus and real-world applications are linked together by chance alone. My goal is to present mathematics through science. Therefore an emphasis is placed on mastering the scientific method of analysis through understanding the necessary concepts of differential and integral Calculus.

My book begins with a brief yet important review of the number system, laws of arithmetic, and some algebra. I have used simple pictures to show the student these fundamental concepts and operations. My definition of a number culminates with a philosophical look at the difference between actions and objects and then explains how a number can represent either attribute. There are no rigorous definition muddled with Greek language and abstract symbols.

The goal is for the student to visually understand the operations he or she takes for granted though uses so extensively. It is not obvious why multiplication and division dominate most fundamental equations from engineering to biology. For example many students can not explain conceptually the difference between 2+2 and 2 X 2, much less why a fraction times another fraction results in smaller number. This is due to the student’s inability to relate the concept of the number with multiplication. The chapter I have written reduces all the operations of arithmetic with the philosophical definition I have given to the number as an independent entity.

After this chapter I begin my in-depth study of Calculus with an introduction to the function, giving full and complete definitions that center on perceiving it as a mathematical relationship. I introduce the concept of dimension to show how a function defines a situation in terms of its interacting conditions or dimensions.

I then show how it is possible to graph a function in a two-dimensional system using the idea of the independent and dependent variable. From plotting a few points to plotting more points I intuitively show the behavior of a function is only as accurate the numbers of points it has been evaluated at. Once again it is not obvious how a function can be graphed as continuous changing line. I then give a short proof, using the idea of a small change in x producing a small change in f(x) to explain how the graph of the function is the continuous line drawn through these points. This is all done with graphical examples followed by complete of what exactly is going on.

Having completed the chapter on functions and their graphs, I begin a thorough immersion into the analysis of the function and its graph. Beginning with a few pages just on the definition of rate of change, I move on to a long study of the graph of a function that is not constant. Of course I do not begin with the assumption that the slope is not constant. I divide the graph over an interval into small intervals Δx, over which the graph will be analyzed independently. Again using the idea of slopes I introduce the terms average and approximate rates of change. It is here where I slowly the equation for the average rate of change through any two points on the graph:

$$\frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Having established this equation I then proceed to analyze the graph in a more complete way by breaking it up into intervals of value Δ x equal to a small number. The goal is for the student to understand that we are now looking for a way of defining the instantaneous rate of change of the function. From there I develop the concept of taking the limit and showing exactly what is meant by tangent to a graph. Building upon the slope equation and my definition of the graph of a function as a line connecting a set of points I show what exactly is meant by having a unique line through "one" point. Once again I use clear and simple examples that contain a minimum of notation and focus on understanding the relation between instantaneous rate of change and Calculus.

I then go on to derive the normal rules for differentiation, clearly explaining each step in the derivations. I take considerable time with the chain rule which derives itself from the definition of a function and the rules for differentiating them.

From there I move on to the graphical interpretation of the derivative and the second derivative. As always I go through a clear and complete explanations of concavity and ter 2-Numberpoints by assuming nothing. The underlying goal of this chapter is for the student to understand the relationship between the function and its derivative. This is done by emphasizing that the behavior of a function can be entirely predicted just by studying its derivative, *f'(x).*

Having developed this link between function and derivative I take a more numerical and thorough look at how the derivative alone can be used to accurately graph the original functions. First I show how the equation:

$$\frac{df}{dx} = f'(x)$$

can be thought of as $\Delta f = f'(x)\Delta x$ I take considerable time explaining the difference between the discrete Δx and the differential . Using this new equation I explain linearization and errors. In the next section of this chapter I introduce the process of integration. By taking a graph $f = x^n$ and breaking it up into four continuous sub-intervals I show how the exact change in the function, f, over the entire interval can be found by evaluating the *f'(x)* Δx over each sub-interval Δ *x* and then summing up the *delta x f* . In the discussion it becomes rather obvious how the exact δ f's of the graph can only be found by letting ou*r* Δ *x* go to a smaller number and summing up the many *f'(x)* Δ *x* over the interval. In explaining all this I only use two new symbols:

$$\sum_{x=a}^{x=b} f'(x)\Delta x = \Big[_a^b f(x)$$

It is in my highly theoretical chapter on Integration do I delve into the meaning of this equation and introduce the integral sign between the two symbols. What I do in this chapter is basically undo all that I did with the derivative. Throughout my discussion of the derivative I show it as a subtracting and dividing process. In explaining integration I show it as a multiplying and adding process or the reverse of differentiation. The emphasis is placed on understanding the relation the derivative has to its integral in terms of the net change in a function f(x) from a to b can be found by evaluating the derivative times a small Δ x and summing up these values from a to b. I show intuitively how the error gets less and less as Δ x gets smaller by using areas. By presenting different simple proofs for the Fundamental theorem of Calculus, there should be little doubt in the students mind as to what integration means and more importantly how it relates to concept of differential change.

The next chapter I go through a series of geometric applications from areas to arc length. As an Engineer I have seen that students are generally completely helpless when it comes to setting up integrals. This is entirely due to the fact that the student has no idea what it means to integrate a function although they are more than capable of integrating some of the most awkward looking functions. What this chapter does is draw the bridge between the theory in the previous chapter and the practical applications in the following chapter. Even though the students understands integration, applying it requires yet another theoretical chapter. The transition from *f, x, F, M, P etc..* to $\int \frac{T \cdot \Delta L}{J \cdot R(L)}$ (the integral for finding the total angle of twist of a rod, length L, of varying radius with J as the polar moment of inertia subjected by a twisting torsion force, T) appear to have little to do with each other except that they are evaluated the same way.

In going through the geometric applications I leave behind all the x's, f's and f(x)'s and replace them with their geometric equivalents of length, l, width, w, and height, h. In order to understand how to set up integral I use the concept of a independent and dependent variable to explain when one constant is changing with respect to another constant a differential needs to be written. I do not delve to much into this theory of writing differentials as I leave it to the next chapter where identifying the differential is not a simple matter of $\Delta A = h(w)\Delta w$. I concentrate on understanding the integral as a three step process of summation, integration and then evaluation. What is important is the method as the previous chapter already explained how these processes are all related.

It is in the final chapter on applications of integration that it becomes necessary to develop a consistent theory of integration that can be applied to any problem from engineering to economics. I further develop the idea of a constant being expressed as a function of another constant and then explaining how to identify the independent variable in the equation. I present a logical and reasonable algorithm for setting up integrals in problems.

This is perhaps the most important chapter in my book as its goal is to reconcile the theory of function, change, differentiation and integration all into one clear and concise method. This is really what Calculus is all about. Modern textbooks devote most of their time to evaluating integrals while they completely disregard the numerous steps that must be taken before an integral can be set up and evaluated. A student must be able to understand how to set up an integral in a practical situation be it from electromagnetic forces to dynamic response of a skyscraper during an earthquake. Calculus is an utterly useless tool without this fundamental understanding of what integration is all about as the student will be able to play with Calculus but he or she will never know how to use it.

This ends my text-book on Calculus. My book is intended to offer an introduction to Calculus for College or High-School. The course may be completed in either one semester or one year, depending how in depth the instructor is prepared to go. Throughout this Preface or introduction, I have constantly been referring to my complete discussions and explanations that are unlike the nonsense that accompanies theorems and concepts in other books. It is difficult to explain how exactly it is different for that would require my re-writing the book in these opening line. I can assure you that what awaits you is something unlike anything you have read and studied before.

More than 85% of the work is original and entirely conceived using the power of reason and logic. The remaining 15% is material that is covered in most every text. Though my presentation of such material has resemblance of form, in terms of content it is considerably different. Other books present a puzzle whereas I have taken the time to complete that puzzle not to mention adding further observations along the way to help strengthen the students understanding.

It is therefore no overstatement to say that the entire book is unique and has almost no connection to the modern textbook. The only way to judge the validity of this statement is to continue on and read it. I am certain what you will learn will fascinate you.

Chapter 1 - Why Study Calculus?

Why study Calculus? As a student you probably view Calculus as another illogical memorization of equations that one needs to pass through school. You feel you will never be called upon to use any of the 'hypothetical gibberish' that you learn. Consequently, you approach the subject with philosophical indifference and adapt yourself to endure that which you can not cure.

Perhaps somewhere behind this ill-sentiment is the belief that Calculus is a necessary tool for your existence in the real-world. Regardless of how one defines a successful existence, an understanding of Calculus is **not** essential to your well-being. In fact it is an overstatement to claim that one needs it to become even a scientist. After all, many professional engineers do not hesitate to admit their ignorance of the subject. Such engineer's understanding of Calculus is limited to knowing the equations and knowing how to use them. Calculus is thus reduced to a few formulas that are blindly used for number crunching.

So far this introduction has only succeeded in confirming what you may already feel about your study of Calculus, i.e it's a meaningless waste of time! Why do you have to be bothered learning it while others get by with absolutely no understanding of it? What more can there be Calculus apart from memorization, manipulation and frustration? Without fueling your growing suspicions any further, let us return to the original question and answer it from a purely non-materialistic point of view.

Why study Calculus? This questions is best answered with another question; What is Calculus about? It is extremely difficult to answer this briefly and convincingly for that would require paraphrasing this entire book into just a few lines of text. I could present you with a series of situations where Calculus is applied but that would give you the false impression that practical applications and mathematical theory are linked together by chance alone. The simplest definition I can offer is that Calculus is the study of mathematically defined change. There are two words in that definition that require further explanation before it can make sense. These words are **mathematics** and **change** .

What is mathematics? Essentially mathematics is nothing more than the language of science. While science is a systematic study of nature, mathematics is a concise form of communication used to represent nature. Man's faculty of reason allow him to observe, dissect and hypothesize nature and all its ongoing process's such that the end result of this orderly analysis is mathematics. But what is meant by science and nature? Rather than continue with questions answered only by generalizations, let us begin this study of Calculus with a study of Man.

What makes man act, think, feel, move and function? The subject of philosophy is devoted entirely to answering questions like these, though it often asks more than it is prepared to answer. Life around us constantly challenges us to live and respond to the seemingly impossible and unfathomable. We are confronted with dilemmas that require our judgement yet it is this judgement that fails us. The code of ethics and morals we live by turn out to be nothing more than a set of ironic contradictions. We find ourselves surrounded by a sea of chaos where the only way to prevent ourselves from drowning is to clutch the nearest piece of wood from the wreck. Life then carries us thorough the sea of confusion, never stopping anywhere to drop us at a destination. Those who look to the sky only find yet another wide expanse of unconquerable area. Life appears futile with no meaning in sight.

Is man confined to this helpless state of existence? To be honest, no one knows the answer. We can, however, look for hope in the mind. The power of thought, reason and logic are what allow man to seize control of life around and ultimately serve as a guide to enlightenment and wisdom. Aristotle felt that happiness could only be achieved by cultivating the mind. He wrote:

> *Now the peculiar excellence of man is his power of thought; it is by this faculty that he surpasses and rules other forms of life; and as the growth of this faculty has given him supremacy, so, we may presume, its development will give him fullfilment and happiness.*

This leads to the question, *How do we define reason?.* This is synonymous with asking how do thoughts define behavour? As a child you had little control over your actions. There did not seem to be any purpose behind what you did or what happened to you. Life carried you along, giving you the freedom to enjoy the ride.

The idea of free-will offers a more plausible explanation of human actions. It says that man has the **freedom** of choice to decide on a certain course of action. Actions are the results of choices; choices which we are **free** to make.

The branch of philosophy known as determinism takes free-will a step further to explain why man chooses to act in one way over the another. When man is confronted with a situation, the line of action he decided on taking is based on experience, personal interests and preferences. The decisions we make are thus entirely influenced by our past experiences. The will is **not** free to behave on its own. For example my decision to write this book was not based on some impulsive instinct but on a culmination of circumstances I was exposed to and reacted against.

Any action from the most random to the most perverse can be explained by the set of situations, experiences, thoughts and feelings that preceeded that action. Essentially determinism says that our lives are pre-determined to the extent that regardless of how we live we can never *change* our course of life. As a French thinker once said , *We change yet we remain the same".* The logic behind this statement is that if our past can never be changed than our future will always remain the same. Each action is dictated by the previous one.

It is at this point where science and art diverge. For the scientist, determinism is an accurate enough explanation of human life, as it says that everything that occurs, occurs for a set of reasons. Observing and understanding these set of reasons becomes the work of the scientist. The artist, however, interprets determinism as saying, 'Since everything in life is pre-determined then life is meaningless'.

Life may have no meaning but it is the goal of the artist to question that statement by exploring the mysterious depths of human nature and the heart. Perhaps it is the utter randomness of life that causes us to ignore our unchanging destiny. No human being has any control over actions that he or she will be exposed to. Fate begins to lose meaning as one never knows what **will** happen to oneself. Wisdom seeks chaos to drive its desires.

On the other hand, the study of nature is more precise and is less likely to be influenced by a wide variety of unrelated factors. Within natures all actions, occurances, or changes are dependent on a few factors that can be carefully isolated and studied individually. Science is specifically about analyzing these interacting systems and then forming hypothesis that can accurately explain them.

What makes observing phenomena in nature so interesting is that they always occur in a closed setting where external factors can easily be removed to leave behind just a few interacting objects. It is these objects along with their properties, that become the focus of study. Any attempt to logically explain their unique interaction must come from the objects themselves and not from imaginary external factors. Through reasoning and observation, nature can be understood, such that the future can be determined from the present. As Sherlock Holmes would often warn Watson, *" You see but you do not observe!"* .

Often enough, human beings fail to grasp this simple rule of nature by ignorantly attributing any naturally occuring phenomena to the Gods, heavens, or some mysterious substance with superpowers. Understanding and accepting the truth requires an open and critical mind. Charlotte Bronte humorously wrote about this ironic flaw of human nature in her popular novel, Shirley.

> *Note well! Whenever you present the actual, simple, truth, it is somehow always denounced as a lie: they disown it, cast it off, throw it on the parish; whereas the product of your own imagination, the mere figment, the sheer fiction, is adapted, termed pretty, proper, sweetly natural: the little spurious wretch gets all the comfits - the honest, lawful bantling all the cuffs. Such is the way of the world...".*

Not only is science stymied by ignorance and deception, but it also muddled by the work of the pseudo-scientist. The pseudo-scientist is described by the Spanish philosopher, Jose Ortega, in his stunning book on modern western civilization, The Revolt of the Masses:

> *"... By 1890 a third generation takes command in the intellectual world, and we find a type of scientist without precedent in history. He is a person who knows, of all that a routinely dutiful man must know, only something of one specific science; even of this science, he in well informed only within that limited area in which he is an active researcher. He may even go so far as to claim he an advantage is not cultivating what lies outside his own narrow field, and he may declare that curiousity about general knowledge is the sign of the amateur, the dilettante.*
>
> *Immured within his small area, he succeeds in discovering new facts, advances the science which he scarcely knows, and increases perforce the encyclopedia of knowledge of which he is conscientously ignorant..."*

One of the remarkable masterpieces of the mind is the science of mathematics, often called the science of pure reasoning. While science is a logical system of thought used to study the natural world; mathematics is the precise language of science. It is the **form** of communication for scientific analysis. Number and symbols are nothing more than vague abstractions unless they refer to something specific. Before mathematics can exist there must be a situation to give it meaning. It is scientific analysis that determines the structure of mathematics.

Through mathematics we are able to define the present. The present is only dependent on the conditions that exist within the short frame of time that it occupies. Quickly it vanishes before our eyes, becoming a memory. The goal of science is to define the objective world in terms of existing quantifiable conditions expressed by mathematics. Our dimensions or properties remain fixed and do not change.

It is when our dimensions change that our study becomes a bit more complicated and Calculus arises. But first, what is meant by change? To understand change we need to explain the concept of time. By definition, time is a passage of events, such that for time to pass, something must change with respect to itself. For example a moving object implies a *changing* distance covered from a reference point. This comprises an event that defines time. Or a rising temperature implies that the temperature is changing , thus occupying time. Changes are the results of **actions** that comprise a situation.

While Calculus is the study of mathematically defined change, it is not neccesarily the study of time alone. In science other dimensions can be changing with respect to each other. For example velocity can change with height, temperature change with energy, density change with depth, force change with mass etc.

When a dimension is changing with respect to itself, we say it is changing with respect to time. When factors change with respect to each other, we disregard the effect time has on the factors and proceed to only analyze the interacting dimensions. We assume our factors are constant that change with respect to each other, not with time. Calculus is thus the branch of mathematics used to study any phenomena involving **change** . Change is a relative concept that can involve any pair of dimensions, time, force, mass, length, temperature etc. This may sound a bit abstract but it will become much clearer as we follow through the course.

This concludes the answer to what Calculus is about. You may not feel I answered the original posted question, " Why study Calculus?". The purpose of studying Calculus is simply to introduce your mind to the scientific method of analysis. Through science, practical problems can be identified, explanations generated and logical solutions selected. The aim is for you to understand how to apply your mind in a systematic manner toward understanding the world around you.

Engineering relies more on this fundamental logical approach toward problem solving than it does on sheer number crunching and formula manipulation. For this reason many engineers have forgotten all the theorems of Calculus, but what remains is the important conceptual framework of method and application. Good engineering sense is defined as the ability to quickly identify a problem, come up with practical solutions and then select the most efficient option. Many real-world problems are independent of any complex mathematics, but the same systematic scientific approach is required to solve them. The application of sound logic is all that is required to reduce complexities to simplicities.

Early in the introduction I stated that it was an over-statement to claim that one needs Calculus to become a scientist. Those engineers and scientists who got by with a cursory knowledge of the subject fall into the realm of the pseudo-scientist and charlatan. As Ortega wrote, the pseudo-scientist, "advances the science which he scarcely knows, and increases perforce the encyclopedia of knowledge of which he is conscientously ignorant...". It is the thrill and enlightenment of understanding nature that drives the scientist. The engineer takes science a step further to control nature to suit man's needs. Both the engineer and scientist share a deep appreciation for the workings of nature; an appreciation that develops into wisdom.

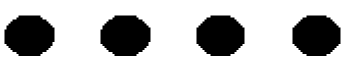
Therefore, the purpose of studying Calculus is two-fold. First to introduce you to the basic concepts of mathematics used to study almost any type of changing phenomena within a controlled setting. Second, studying Calculus will develop invaluable scientific sense and practical engineering problem solving skills in you. You will understand how to think logically to reduce even the most complex systems to a few interacting components. As you study the main concepts, theories and examples in this book, your mind will develop into a powerful systematic instrument.

Questions

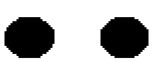
- Has God created man or man created God? And if man created God does that mean he exists? Comment on this question with regards to your views on religion and society.
- The mind constantly struggles to balance the emotional contrast between boredom and activity, pleasure and pain, happiness and sadness, etc. In fact the balance is so complete that one can claim that the net sum of all feelings is zero. Do you agree? Think about what makes you happy, content or sad and compare it with what made you happy, content and sad five years ago.
- What is good and evil? Are morals absolute or do they exist only relative to each other? In other words do you think certain actions can be **judged** as pure good or evil, regardless of personal upbringing?
- Make a list of activities you **enjoy** doing and a list of things you **hate** doing. Make it in a grid fashion so that your classmates can rate each activity from 0-10, with a 10 meaning they share your preference, while a 0 means they feel exactly the opposite way you do. Be as specific as possible with regards to your daily life. The results should teach you something about human nature and values.

Chapter 2 - Numbers and their Uses

What is a number? What do all those abstract symbols floating about really mean? To begin with, we can define the number as an existence. So if I had a few tennis balls, the number of balls would simply refer to how many individual tennis balls I am actually holding. 4 balls would mean four existences or :



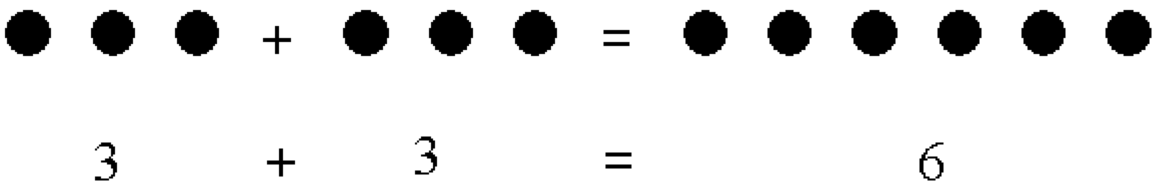
2 balls would mean



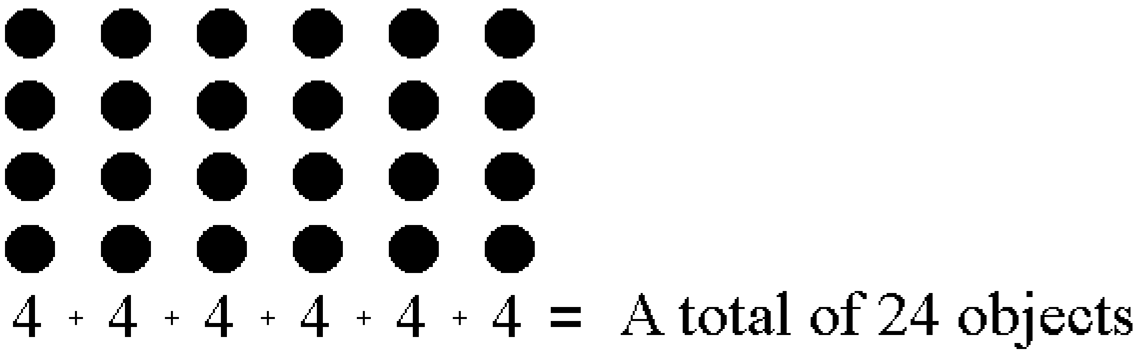
A number therefore does not depend on any specific quality of the object. 10 balls and 10 eggs are the same number of things.

Now that we have defined the number as something created by the human mind to count objects, then let us proceed to examine the Laws of Arithmetic.

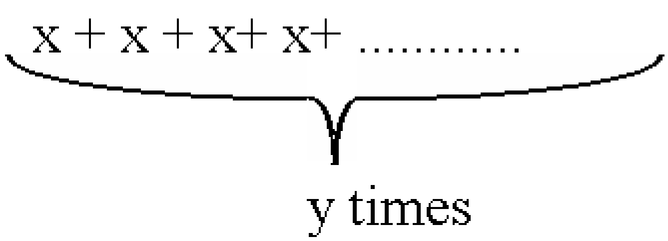
The laws of arithmetic are based on the addition and multiplication of numbers. But what does it mean to add two or three numbers together? To begin with addition assumes that all your numbers refer to something and that thing is the same throughout. So 3 pens added with 3 pens is six pens. We are bringing together all our things and then counting the sum. So:



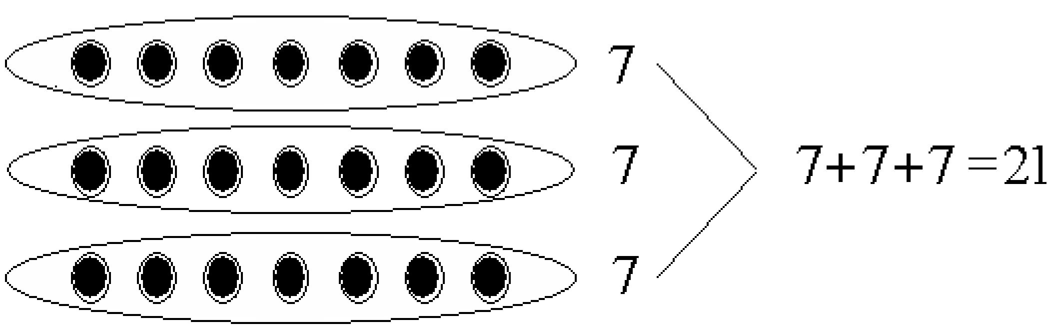
In multiplication we are doing something a bit different, we are creating multiples of a number. When we multiply two numbers x and y : we are creating y multiples of x which is the same as adding x with itself y times. This can also be thought of as a grid of x rows and y columns where the product of the numbers is the sum of all the objects inside the grid. So 4 * 6 is:



In this example we only added 4 with itself 6 times and summed up the number we got. This is all that multiplication is about, creating multiples of an number or just repetitive addition. Multiplication of two numbers x and y can therefore be expressed as:



or 3*7 is: 7 + 7 + 7



You are probably asking what is the use of such an operation or why is multiplying numbers important an important thing in mathematics. It is fairly obvious how it used when a shopkeeper wants to find the total cost of 72 oranges at $3/orange. The answer is 72*3. But is not so easy to understand for example why in Newton's second Law of Motion:

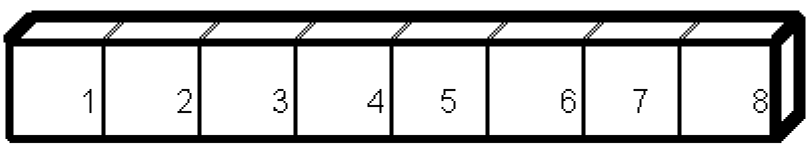
$Force = Mass * Acceleration$

or why interest earned in money is:

$Interest = Principal * Rate * Time$

The reason multiplication dominates these equations and so many others depends entirely on the context of the equation and where it is derived from. As we study many equations in this book the difference between multiplication and addition will become clearer.

We have studied Integers which refer to full existence's such as 2 men, 4 girls, 11 donkeys and 110 bananas. But what about half a candy bar or a third a day? When the existence or object is broken up into equal parts we say we are dividing and we use fractions to represent the division. Division is essentially the opposite or inverse operation of multiplication. Whereas multiplication of creates equal sets of a number, division creates equal sub-sets within the number. The fractions that are used to represent division are often written as: $\frac{1}{2}, \frac{1}{7}, \frac{2}{3}$ where the bottom number refers to how many pieces the top number has been broken into. For example the following rod has been broken into 8 partitions where each small block is: $\frac{1}{8}$'th of the original object:

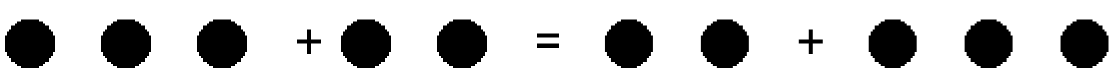


We can now define x divided by y to be $\frac{x}{y}$ where y indicates the number of equal parts an object is subdivided into and x indicates how many parts you have. The relationship between multiplication and division is best expressed by the equation, $\frac{1}{y} \cdot y = 1$ If there is an x on top of the y it tells us that there are x times $\frac{1}{y}$ or x parts of $\frac{1}{y}$

Whereas division creates sub-sets within an object, multiplication creates sets outside the object. Dividing an object by another number x, creates x sub-sets within the object such that multiplying one sub-set by x give the original object again. For this reason multiplication and division are called inverse operation of each other.

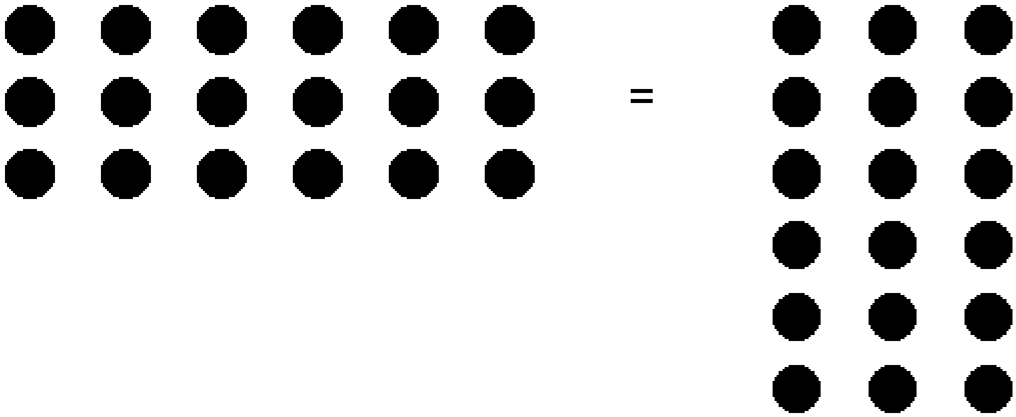
We can now state the five laws of arithmetic:

1) A + B = B + A



2) A + (B + C)= (A + B) + C

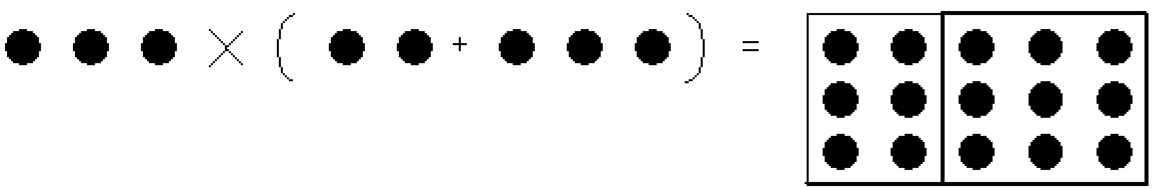
3) AB = BA



4) A(BC) = (AB)C

The fourth property is saying that multiplication of more than two numbers is independent of the order they are multiplied in.

5) A(B + C)= AB + AC



To this point our discussion of numbers has been limited to positive numbers but what about negative numbers? Before discussing what it means to have -3 CD's let us look at the number 0. The use of zero in the number system did not appear till after the Greco- Roman period with the Arabs. Just as numbers define an amount of something, so we must have a way of defining nothing. Zero, therefore represents nothing or no-existence. It is just as much a number as 6 or 7 is. For example:

$$7+0=7$$

$$3-3=0$$

$$x+2=0$$

As we shall study, zero is an important number as it helps in solving equations and defining infinity.

Negative numbers, in the simplest sense, are used to define those numbers that are the result of a larger number being subtracted from a smaller one. Imagine a man who has six cigarettes. A young man comes to buy eight cigarettes. How many are left? The answer is not 0 but -2. The shopkeeper sells all six, but he still needs two to give to the man. Negative numbers can therefore be thought of as numbers that do not exist but should exist. They become positive when they take on a physical existence otherwise they are missing pieces in a puzzle.

Our discussion of numbers has been limited to objects. This need not be the case. Numbers can be used to represent abstract existence's such as 2 days, 3 years, 4 classes, etc. Recall how we defined the number to be meaningless without referring to something. Three LP's meant one LP + one LP + one LP. The number can also be thought of as a repetition of an occurrence. A day is defined as the rising of the sun, setting, then rising again. This " action" is cyclical and continues for eternity. We can therefore refer to each cycle of the sun rising as one day and four days would then represent four cycles of the sun rising.

Here we have extended the definition of the number to not only include objects but also actions. The difference between an action and an object is a difficult one to explain. While an object represents something we can see, touch, hold and feel, an action is only that which happens. Actions refer to change or movement. Therefore an actions existence is defined by what takes place during that time frame of a beginning and an end.

Numbers are not limited to objects and actions alone, they can also be used to count ideas and other abstract concepts that exist only in the mind. Simply by existing in the mind means it can be counted. When somebody says I have two things to say to you, each thing is made up of a different choice of words, phrases, not to mention the action or object it may refer to.

To conclude our discussion of the number here is a short definition.

Number - A symbol used to express an objects' or actions' repetitive nature.

Questions

Your job is to set up a 5 star hotel from scratch that will yield maximum profits. Some factors to consider are construction costs, opportunity costs, time-value of money, interest rates, number of rooms, number of staff, hourly wages, utilities, taxes, promotions, etc. It is up to you to make this as complicated as you wish.

Chapter 3 - The Mathematical function and its Graph

Section 3.1 - The Scientific Method

Science is a rational method of observing situations and then forming hypothesis to understand them. The language of science is mathematics and it is through the mathematical function that a situation can be expressed in terms of the conditions that define it. The function is thus a fundamental concept in mathematics and Calculus is the branch of mathematics dealing with functions whose dimensions change. What then is a function and how is it expressed mathematically?

Before we can learn what a function is we need to understand how mathematics relates to science. The **scientific method** is an orderly and efficient way of analyzing physical situations. Its counterpoint, the **trial and error method**, is something you are probably more familiar with. Human nature inclines the mind more toward exploration and experimentation rather than systematic analysis. The trial and error method reflects the tangents that human logic takes when curiosity beckons.

The scientific method, on the other hand, provides an orderly path for the mind to follow in its passage from bewilderment to enlightenment. Without following the scientific method, an analysis will remain consistently complex and any theories formed will be too general to explain the phenomena accurately. Therefore an understanding of the scientific method is crucial to your understanding of mathematics and development as a scientist. Here are its main steps.

- Identify the problem or situation
- Narrow down the problem statement while being as specific as possible.
- Remove external factors to leave behind a few interacting conditions.
- Analyze remaining conditions along with their properties with respect to the entire system.
- Make reasonable assumptions about the controlled situation, such as mass/ energy/ people balance etc.
- Understand how the conditions come together to define the situation.
- Determine a logical relationship among the conditions.
- Select the most efficient solution that accurately explains the system through the interacting conditions.

Engineering and science both follow the same systematic analysis to solve problems and understand the world. Science deals more with objective and idealized situations while engineering is often concerned with subjective situations that require sound judgment. Situations are not always defined by fixed quantifiable conditions. The importance of the scientific method is its emphasis on clear, logical, and focused thinking that can analyze a specific situation in relation to the larger system that it is part of. The method is the quickest cure for frustration and desperation.

Mathematics is a reflection of the orderly analysis and conclusions of the scientific method. It is the language of science used to communicate ideas, theories and observations in a concise manner. From the scientific method, notice how the word, condition, appeared often. In mathematics condition are refereed to as dimensions. The following section will take a closer look at what a dimension represents.

Section 3.2 - Dimensions

The mathematical function is a relationship that defines a **situation** in terms of a set of interacting **conditions**. Neither the situation nor the condition can exist independent of each other. The function states how a set of fixed conditions come together to **define** a situation. In mathematics, conditions are denoted by the term, dimension. Before one can understand the concept of the function, we need to explain what a dimension is.

In science there are four fundamental dimensions used to describe any physical situation in the natural world. They are mass , length,temperature, and time. While mass, length and temperature refer to static situations, time is a dynamic dimension used to describe changes that **define** an **action**.

Think of time as a passage of events. For example a summer spent doing nothing but eating, sleeping and watching TV would seem to have gone by in a few days. On the other hand a summer spent traveling around the world would seem to have lasted forever. If we assume each vacation to have been 80 days, ones memory of the uneventful vacation consists only of a few events that could have been done in a day whereas the vacation around the world consisted of a plethora of events, such that relative to the boring summer, they took longer to finish.

For time to pass something must change with respect to itself. The units of time, the second, is nothing more than a reference for any change. If one second refers to a pendulum's swing through one arc, then *any changing dimension* can be measured relative to the standard of one second. For example a moving car's distance (length) from a reference point is constantly changing. The changing distance can only be measured with respect to time.

It is important to understand that time is the dimension for actions, while length, mass and temperature refer to static conditions. If length, mass or temperature were changing with respect to **itself** then the change would have to be analyzed with respect to time. To summarize, for time to pass some action must occur and for an action to occur some dimension must be **changing** with respect to itself.

Length, mass, temperature, and time refer to the simplest absolute dimensions the physical world can be reduced to. Other dimensions **derived** from these fundamental dimensions include, force (dependent on mass), stress (dependent on force), elasticity (dependent on force and stress), velocity (dependent on length and time), kinetic energy (dependent on mass and velocity), etc.

Many derived dimensions are dependent on other derived dimensions such as kinetic energy is dependent on velocity which is dependent on time. But what about people, dollars, etc. Clearly these are also measurable conditions; however, they can not be easily expressed in terms of the fundamental dimensions of mass, length, time, and temperature. Since they are *unique* dimensions we need to come up with a consistent definition for a dimension that applies to all fundamental, derived, and unique dimensions.

A dimension is simply a quantifiable condition that describes a situation. In science we always study situations where external factors are removed from the analysis. It is the objects that remain along with their properties that become our focus of study. A dimension is a measurement of a property that is relevant to the situation being analyzed. By themselves, dimensions are meaningless. They must refer to certain conditions specific to the situation being studied. One can not refer to just mass or length. One has to specify mass and length of what part of the system? The units of the various dimensions, meter, seconds, Celsius, kg, serves as standards to measure the dimension relative to.

Understanding which dimensions to include in your scientific analysis all depends on the type of situation being studied. The more familiar you are with the fundamental and derived dimensions, the easier it will be for you to understand which dimensions are significant and how they define the situation.

Section 3.3 The Mathematical Function

The mathematical function expresses the relationship between a situation and the conditions that define it. The form of the mathematical function is thus:

$$situation(conditions) = interacting\ conditions$$

This is read as, a situation is a **function** of certain conditions*. The conditions are called dimensions*. For example consider the mathematical function for calculating the force acting on a body of mass, m.

*Force = Mass * Acceleration*

The force on a body is a situation defined by two conditions, mass and acceleration. In a closed system , the force acting on an accelerating mass, is the product of the mass and its acceleration. Thus force is a function of two dimensions, mass and acceleration while force is the third dimension dependent on m and a of the system. Therefore force is called the **dependent dimension** while mass and acceleration are the **independent dimensions** that together define the dependent dimension. The dependent dimension, F, is placed outside the brackets, while the independent dimension are inside the brackets. Remember . the language of mathematics is as concise as possible. Thus the function is more commonly written as:

$$F(m,a) = m \cdot a$$
$$F = m \cdot a$$

For example the force required to accelerate a 5 kg object at 3 m/s/s is:

$$F(5,3) = 5 \cdot 3$$
$$F = 15N$$

One is free to use any value of m and a to calculate the corresponding force. For this reason, m and a are called independent dimensions, while F is the dependent dimension defined by them. Note that m and a must come from the **same** system, and the calculated force refers to that situation only.

What if I were given the acceleration of a body and the force acting on it and wanted to calculate its mass? In other words, given F and a, what is m? This is done by solving the force function for m or:

$$F(m,a) = m \cdot a$$
$$F = m \cdot a$$
$$m = \frac{F}{a}$$
$$m(F,a) = \left(\frac{F}{a}\right)$$

Therefore, mass as a function of its acceleration and the force acting on it, is the force divided by the acceleration. Here mass is the dependent dimension dependent on force and acceleration. Once again, the three dimensions, mass, force, and acceleration must all refer to the same situation.

It is not uncommon in engineering to encounter twelve to fifteen dimensional functions. For example consider the five dimensional function for calculating the elongation of a cable being stretched with some tensional force.

$$Elongation(Tension, Length, Area, Elasticity) = \frac{Tension \cdot Length}{Area \cdot Elasticity}$$

$$E = \frac{T \cdot L}{A \cdot E}$$

The total elongation of the cable is dependent on four conditions. The tension acting on it, its length, the cross-sectional area, and a material property called elasticity. The point to understand is that the n'th dimension is entirely dependent on the other (n-1) independent dimensions. The (n-1) independent dimensions represent a set of fixed or constant conditions that together define the n'th dimension

Questions?

- The distance covered by a falling object near the surface of the earth is given by:

  $$distance(acceleration, time) = \left(\frac{1}{2}\right) \cdot acceleration \cdot time^2$$

  Simplify this function using d for distance, t for time etc.

  The acceleration can assumed to be a constant 9.8 m/s/s. Substitute this constant into the function and write the expression for distance as a function of time.

  Tofind approximately how tall you are, drop an object from the top of your head and using a stopwatch, find the time taken for it to hit the ground. Substitute this time into the distance function just derived. The answer should be close to your height. Try this at least ten times to get a close approximation.

  Calculate your height in feet using the following function:

$$feet(meters) = 3.281 \cdot meters$$

$$f = 3.281m$$

Section 3.4 - Two dimensional functions

Calculus is about analyzing functions that change. Until now it has been assumed that all the independent dimensions in an n-dimensional functions are **constant** and remain fixed. It is not difficult to see how changes in an independent dimension give rise to changes in the dependent dimension. For example if the mass of an object doubles, the force to push it at the same acceleration will also double.

Allowing more than one dimension to changes poses a challenge that is the focus of an advanced course, multi-variable Calculus. The scope of this text will be limited to n-dimensional functions in which only **one** dimension is allowed to change. The n-dimensional functions thus reduces to two dimensions, i n which a change in the independent dimension produces a **proportional** change in the dependent dimension.

Before explaining what is meant by a changing dimension, let us see how functions of several dimensions can be reduced to simple two dimensional functions. Consider the four dimensional function for the attractive force (gravity) between two bodies of different masses separated by a certain distance from each other. The force of attraction is given by Newton's Law of Gravitation:

$$Force_{gravity}\left(mass_{body1}, mass_{body2}, \text{distance}\right) = G\frac{mass_{body1} \cdot mass_{body2}}{\text{distance}^2}$$

$$F = G\frac{m_1 \cdot m_2}{d^2}$$

$$G = \text{gravitatio nal constant}$$

IF we were given the masses of two bodies and the distance separating them, then the force of attraction would be a simple matter of plug and chug. Suppose we were only given the mass of object 1 as 1000 kg, and the distance between it and object 2 to be equal to 1,000,000 meters. The function would reduce to:

$$F(m_1, m_2, \text{d}) = G\frac{m_1 \cdot m_2}{\text{d}^2}$$

$$F\left(m_1, 1000, 1\text{X}10^6\right) = G\frac{1000 \cdot m_2}{\left(1X10^6\right)^2}$$

The fixed independent dimensions give rise to a constant that can be separated from the function such that:

$$F\left(m_1, 1000, 1\times 10^6\right) = G\frac{m_1 \cdot 1000}{\left(1\times 10^6\right)^2}$$

$$F = c \cdot m_1$$

$\left(\frac{G}{100000}\right)$ was replaced by a constant c. The gravitation force is now only a function of the first object mass, since all other conditions are given. Thus:

$$F(m_1) = c \cdot m_1$$

The force depends on whatever the mass we choose to enter multiplied by some constant that is dependent o the situation already known. Thus we have reduced a four dimensional function to two dimensions by entering conditions that are already known. Therefore, any n-dimenional function can be reduced to two dimensions, provided (n-2) dimensions are already given/

Questions

1) The amount of interest earned by money in a savings account is given by the function:

$$Interest(Money, Interest\ rate, Time) = \frac{Money \times Interest\ rate \times Time}{100}$$

$$I(M, R, T) = \frac{M \cdot R \cdot T}{100}$$

Or simply:

$$I = \frac{M \cdot R \cdot T}{100}$$

1) Given a deposit of $1000 and an interest rate of 6%, write a two dimensional function of interest earned as a function of time

2) Find the interest earned after 2, 3 and 10 years

Section 3.5 - The Graph of a Function

When the independent dimension is free to take on any value, the dependent dimension will depend on what that value is. As the independent dimension **changes**, the dependent dimension will change accordingly. For example the function for the total kinetic energy (energy of motion) of a moving object is expressed in terms of its mass and velocity

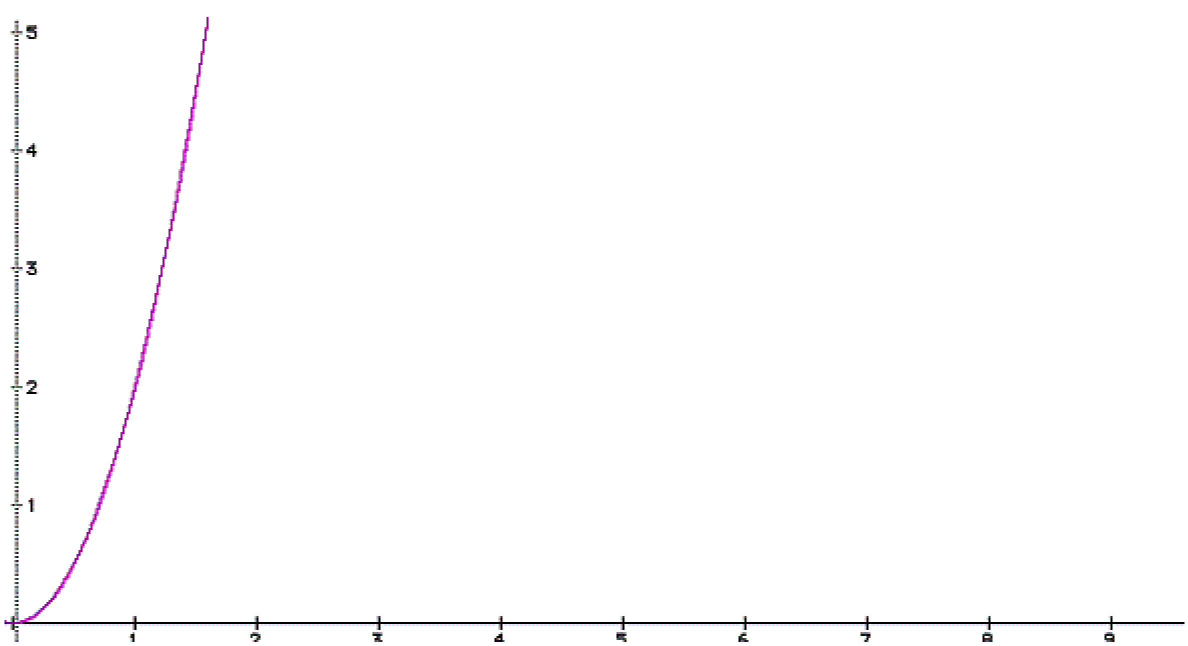
$$E(m, v) = \frac{1}{2} m \cdot v^2$$
$$E = \frac{1}{2} m \cdot v^2$$

Using a constant mass of 4 kg, the function reduces to the following two-dimensional function.

$$E(4, v) = \frac{1}{2} 4 \cdot v^2$$
$$E(v) = 2 \cdot v^2$$
$$E = 2 \cdot v^2$$

The total energy of the body is now entirely dependent only on the square of its velocity, whatever it may be. If the velocity were 1 m/s, the energy would be 2 Joules; if it were 2 m/s, the energy would be 8 Joules. As the velocity of the body increases, its kinetic energy also increases. This makes sense, since the faster a body is moving, the more energy it posses.

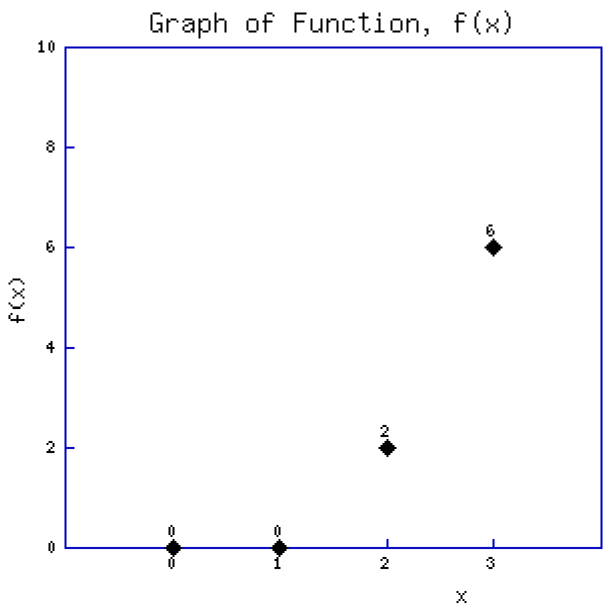
The way to visualize the **changing** relationship between the independent and dependent dimension is by graphing the function on a set of perpendicular axes. The horizontal axis is labeled after the independent dimension (condition), while the vertical axis is labeled after the dependent dimension (situation). The graph of a function is a continuous line that reflect a horizontal change in the independent dimension with a vertical change in the dependent dimension. Each point on the graph represent a unique situation . The graph of the function $E(v) = 2 \cdot v^2$ would be:



It is; however, not obvious as to how the graph of a function is a well-defined continuous line through all the points at which the function exists. To better understand the important relationship between a function and its graph, consider a hypothetical function $f(x) = x^2 - x$. In this function, f represents an imaginary situation dependent on some condition x. Therefore x, is our independent dimension, while f is our dependent dimension.

For the next few chapters we will only be studying abstract f's and x's. This will allow our study of the mathematical function to be more specialized by avoiding the confusion that arises from dealing with so many, F's, m's, a's, l's, E's etc. It is difficult for the mind to keep track of so many abbreviations, much else understand where they were derived from.

$f(x) = x^2 - x$



With intervals of *1* the difference between each evaluated point of our independent dimension x is *1*. This gives a fairly generalized idea of how the dependent dimension, f, changes. From x=0 to x=1, observe that f is zero at both these points. The conclusion is that the functions value is approximately zero from x=0 to x=1. After x=1 notice that f gets substantially larger, particularly between x=2 and x=3. This leads to the observation that as x increases f gets larger and larger.

Also notice that over each sub-interval the net change in y is different.

From x=0 to x=1, f goes from 0 to 0 or the net change in f is 0

From x=1 to x=2, f goes from 0 to 2 or the net change in f is 2

From x=2 to x=3, f goes from 2 to 6 or the net change in f is 6 - 2 = 4

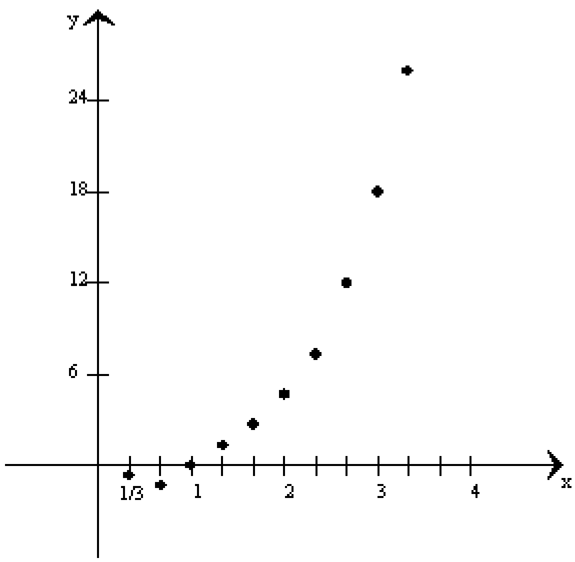
In each case the change in x is 1 but the corresponding change in f increases as x increases.

Until now our discussion of the function $f(x) = x^2 - x$ has been limited to just four points, x= 0,1,2, and 3. The concept to understand is that a function is defined at every point and the continuous line connecting the points corresponds to the graph of the function. In order to gain a better understanding of the function, it needs to be analyzed at more points.

Though before continuing, an important symbol will be introduced, the Greek letter Delta, Δ. Δ is a symbol for a *quantifiable* change, not metaphorically speaking! It represents a definite difference or change between two values. For example if x changed from 8 to 21, instead of saying the change in x is 21-8 or 13, it is more accurate to say Δ x is 13 or Δ x = 13 . This symbol will be used throughout the book to reflect the amount a dimension changes by.

Returning to our example, $f(x) = x^2 - x$, we observed how as x increased, f increased at a greater rate. However, between x=0 and x=1 there appeared to be some inconsistency in the function since its value was 0 at both endpoints. Let us now analyze the function at intervals of x differing by $\frac{1}{3}$ or Δ x = .333. By reducing Δ x from 1 to .33, the function can be analyzed at ten points instead of only four.

The change in our independent dimension x will be .333 and we will be evaluating points at this intervals. Plotting these points gives us:



The pattern from x=1 to x=3 remains the same with f increasing as x increases; however, from, x=0 to x=1 something new appears. Notice how the function seems to decrease until x=.67, and then increases afterward. In analyzing these different graphs (the first graph consisting of only four points) of the same function we can state an important observation.

The behavior of a function as reflected by its graphs is only as accurate as the number of points plotted over the interval of interest.

That should mean if Δ x were infinitely small, then the graph of the function would be the line connecting the infinite amount of points. However, even this does not tell much. Through an infinite number of points we get a more and more accurate look at the functions graph or behavior, but between any two points x and (x + Δ x) where Δ x is the infinitely small distance separating the two points, there exists a further infinite amount of points such that we cannot be sure that our graph takes them into consideration? In other words how do we know if the graph is continuous over the small interval separating the points? Our graph could suddenly jump up or down as it did between x=0 and x=1.

To prove that the graph of a function is continuous over an infinitely small interval consider a simpler function, $f(x) = x^2$ An infinitely small interval implies a change in x, Δ x, equal to nearly zero. Therefore if $f(x) = x^2$ is defined at any value for x, $f(x + \Delta x)$ gives the f value of a point located a distance Δ x from x.

$$f(x) = x^2$$
$$f(x+\Delta x) = (x+\Delta x)^2$$
$$f(x+\Delta x) = x^2 + 2x\Delta x + \Delta x^2$$

If the distance, Δ x between the two points x and (x + Δ x) is allowed to go to zero, what we are doing is *taking the limit* as Δ x goes to a number gets closer and closer to zero. Mathematically this is expressed as:

$$\lim$$
$$\Delta x \to 0$$

Taking this limit:

$$\lim_{\Delta x \to 0} \quad f(x+\Delta x) = x^2 + 2x\Delta x + \Delta x^2$$
$$\lim_{\Delta x \to 0} \quad f(x+\Delta x) = x^2 + 2x(0) + 0^2$$
$$\lim_{\Delta x \to 0} \quad f(x+\Delta x) = x^2 + 0 + 0$$
$$\lim_{\Delta x \to 0} \quad f(x+\Delta x) = x^2 = f(x)$$

On the interval from x to x+ Δ x the function or y-value changes negligibly such that $f(x) \approx f(x+\Delta x)$

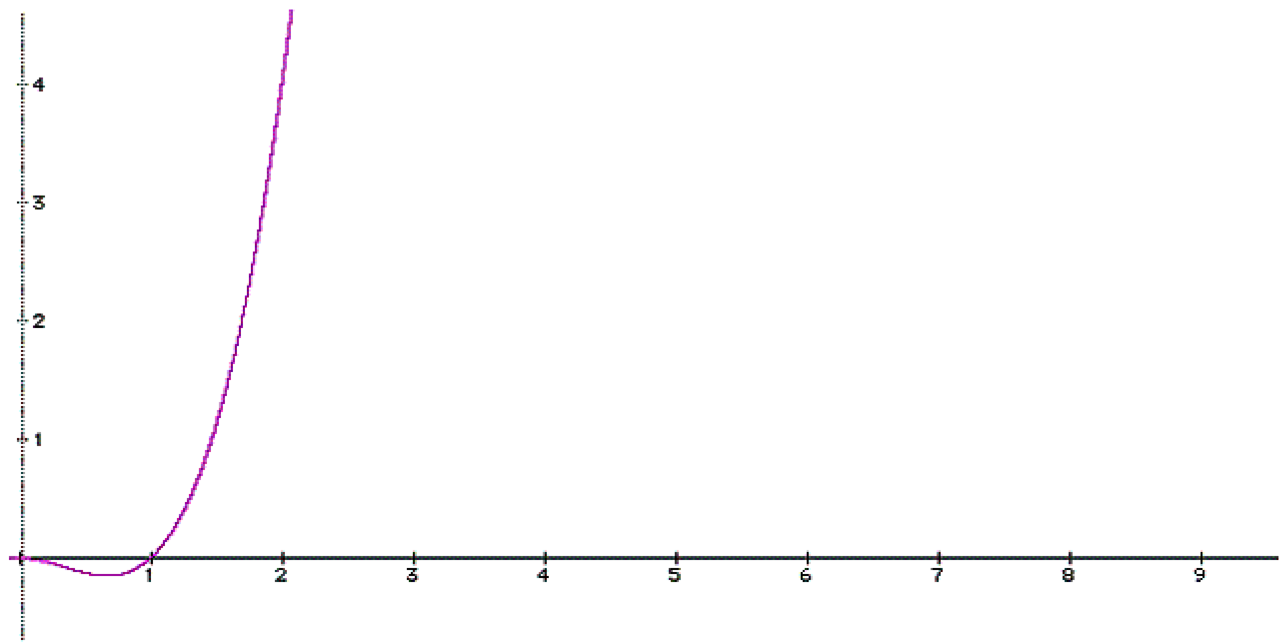
$$x^2 < y(x+\Delta x) < x^2 + 2x\Delta x + \Delta x^2$$
$$x^2 < y(x+\Delta x) < x^2 + \text{An infinitely small number}$$

This proves that points on the graph that are in between two infinitely close points are all close to the values of the endpoints and do not jump up and down in a random fashion. Each successive point in this interval differs from the previous point by an amount that is related to the change in the independent dimension. The graph is uniquely defined as a function of the independent dimension such that for a small Δ x the Δ f is proportionally small.

Intuitively it is not difficult to understand why the graph of a function is continuous over an infinitely small interval. By definition the mathematical function expresses a relationship between dimensions, such that one dimension is said to be dependent on the value of the other dimension. The graph of a function reflects this relationship by relating a horizontal change of the independent dimension with a vertical change in the dependent dimension. Now if an imaginary function $f(x) = x^n$ were to change by an infinitely small amount then the f value would also change a similar minimal amount, because the value of f is **dependent** on the value of x.

The graph of a function is then the continuous line drawn through an infinite set of points. For the graph to be continuous it only needs to be defined at all points where it is being studied.

The graph of $f(x) = x^2 - x$ can now be drawn as a continuous line that corresponds exactly to the behavior of the function, f, as the independent dimension, x, changes:



Questions

Whenever an object is thrown in the air towards some direction, the path it follows is called a trajectory and the study of its motion is the focus of projectile mechanics. The motion of a projectile is dependent on several conditions such as its initial velocity when set in motion, the angle of inclination, the force of gravity, air-resistance etc. What makes projectile motion so interesting is that gravity only restricts the vertical rise of the body, but has no effect on its horizontal motion parallel to the earth's surface. This situation gives rise to the curved trajectory or path of the projectile.

The goal of the following problem is to find the equation for the trajectory (graph) of a ball struck in motion by a bat at an angle of 45 degrees with an initial velocity of 50 m/s. Here is a simple diagram of the situation:

The height of the ball in motion is dependent on the horizontal distance covered and is given by the equation:

$$height(distance) = \tan(\theta) \cdot distance - \frac{gravity}{2 \cdot (velocity_{initial} \cdot Cos(\theta))^2} \cdot distance^2$$

$$h(d) = \tan(\theta) \cdot d - \frac{g}{2 \cdot (v_i \cdot Cos(\theta))^2} \cdot d^2$$

d = horizontal distance

$\mathrm{v_i}$ = initial velocity

θ = initial angle of inclinatio n

g = gravity

Substitute the known values for the given conditions (use g=9.8) to reduce the function to one of two dimensions, involving vertical height as a function of horizontal distance covered.

Geta rough idea of the graph of this projectile motion by plotting points at intervals of Δ d = 1 meter. As soon as the height is negative stop plotting.

Use a Δ d=.5 m, and determine the maximum height reached and the distance from the firing point when it hits the ground.

Use an angle of 60 degrees and see how the graph changes.

Double the initial velocity while using the same 60 degree initial angle of inclination.

Chapter 4 - The Derivative

Section 4.1 - Rate of Change

In the previous chapter we studied the mathematical relationship between an independent and dependent dimension. The relationship expressed in the form of a mathematical function defines a situation for a given set of conditions and properties. Since we allowed the independent dimension to change, it was no longer a fixed quantity but became a variable. The graph of the function reflected a horizontal change of the independent variable with a vertical change in the dependent dimension. This graph allows us to visualize how a situation **changes** with respect to a change in the conditions. This chapter will focus on analyzing graphs and how they represent change.

To find out how much the function, $f(x)$, changes between two points $x_1 \text{ and } x_2$, we simply enter in the two values for the independent variable x and then calculate the difference between the dependent variable, *f*, for those given conditions. Remember that a variable is nothing more than a dimension that is allowed to change or take on any value. Thus, from $x_1 \text{ to } x_2$, the change in the independent variable , referred to as Δx is:

$$\Delta x = x_2 - x_1$$

The corresponding change in the dependent variable , referred to as Δf is:

$\Delta f = f(x_2) - f(x_1)$ Δx refers to an interval over which we are analyzing the change in the dependent dimension, *f*.. The second point x_2 can be written in terms of the first point x_1 plus the change in the variable, Δx

$$x_2 - x_1 = \Delta x$$
$$\therefore \mathbf{x_2 = x_1 + \Delta x}$$

Therefore the change in the dependent variable over an interval from x_1 x_2 $\Delta f = f(x_2) - f(x_1)$ which can also be written as:

$$\Delta f = f(x_1 + \Delta x) - f(x_1)$$
$$\text{where } \Delta x = x_2 - x_1$$

For example the change in the function $f(x) = 3 \cdot x^3$ *x=3* to *x=5* , where Δx is 5 - 3 = 2 is:

$$\Delta f = f(x_1 + \Delta x) - f(x_1)$$
$$\Delta f = f(3 + 2) - f(3)$$
$$\Delta f = 3 \cdot 5^3 - 3 \cdot 3^3 = 375 - 81$$
$$\Delta f = 294$$

Questions

Hooke's law states that the force required to stretch a spring is directly proportional to the amount stretched or

$F(s) = k \cdot s$

F = Force

s = distance stretched

k = spring stiffness constant i.e Force required to stretch spring a unit distance

Graph this function for k=1000 N/m.

Calculate the change in force required to stretch the spring from 1m to 4 m.

The distance a free-falling object covers from an initial dropping point is given by the function:

$d(a,t) = \frac{1}{2} \cdot a \cdot t^2$ *d=distance*

Given that acceleration is equal to gravity or $9.8 \, {}^{m}\!/_{s^2}$, reduce the three dimensional function to one of two-dimensions for time as a function of distance, x.

Graph the function at intervals of $\Delta x = 10m$, from *x=0* to *x=100m*

Tabulate the corresponding change in *x*, Δf , over each interval.

What can you conclude about the change in time covered between x = 10m and x = 20m versus the change in time covered between x = 90 and x = 100m?

Section 4.2- Average Rate of Change

We have learned that a change in the independent variable is defined as $\Delta x = x_2 - x_1$, and the corresponding change in the dependent variable over this interval is $\Delta f = f(x_1 + \Delta x) - f(x_1)$ The question we now must ask ourselves is how can we measure the relative change of the dependent variable with respect to the independent variable? In other words how can we calculate how much more or less Δf changed compared to Δx

To calculate how much more $f(x)$ changed over an interval from x_1 x_2 , we simply divide the change in f over the change in x for the interval. Thus we divide, Δf by the interval over we which are evaluating it, Δx which is equal to $x_2 - x_1$. Thus the relative change of f with respect to x over an interval Δx is defined as:

$$\frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

While this expression may seem rather simple, it does require some explanation. By dividing the change in f by the change in x what we are doing is calculating how much more f changed for a given change in x. For example in the function, $f(x) = 3 \cdot x^3$, when x changed from 3 to 5, f changed from 81 to 375. Over this interval of Δx from x=3 to x=5, the Δf was 294. Thus the relative change in f with respect to a change in the independent variable x is:

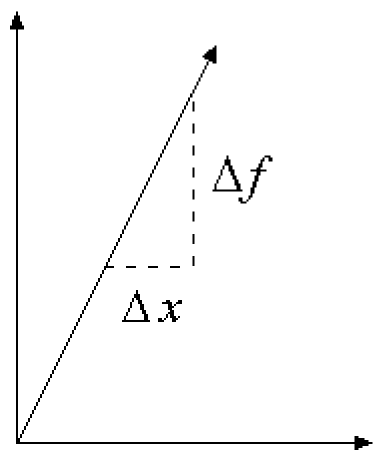
$$\frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

The value of 147 tells us that f changes 147 times faster than x **over that interval of Δx from x=3 to x=5 only.** Thus for each unit change in x, $\Delta x = 1$, the corresponding change in f is 147. We can therefore define the rate of change of a function with respect to its independent variable to be:

$$\frac{\Delta f}{\Delta x} = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

The value, called the rate of change of the function, refers to how much more or less $f(x)$ changes for a unit change in x. Itis only valid is for the interval under consideration, $\Delta x = x_2 - x_1$

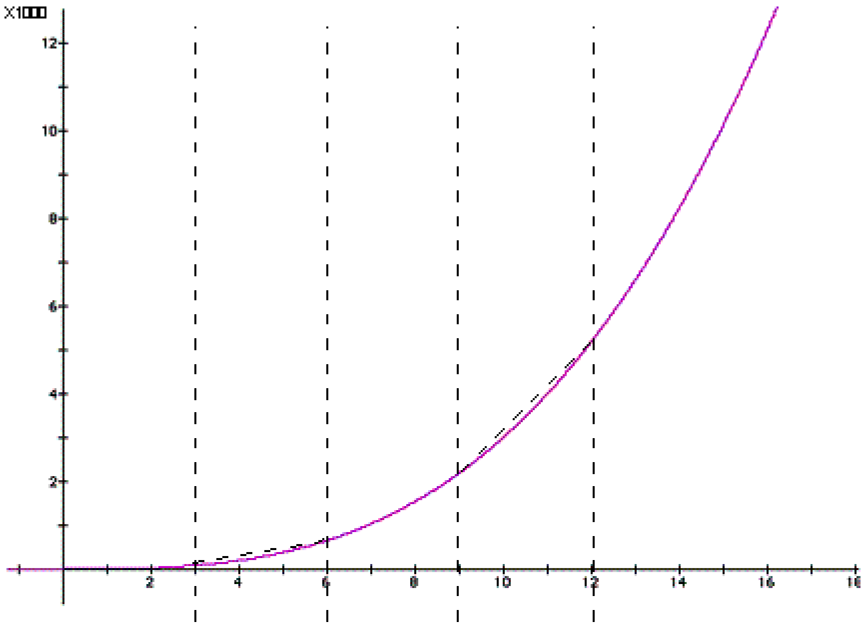
Another way of understanding what rate of change of a function means is to look at the steepness of the line connecting the two endpoints of the interval under consideration.



As Δf increases, the steepness of the line connecting the two endpoints will increase. Thus, the greater the rate of change of the function, the greater its slope or steepness **over the interval under consideration**. Since slope and rate of change are synonymous, then how is rate of change defined for functions whose graphs do not have constant slopes? For example, from x=9 to x=12 of $f(x) = 3 \cdot x^3$, the change in f is 2997. Thus the rate of change of the function over the interval is:

$$\frac{\Delta f}{\Delta x} = \frac{2997}{3} = 999$$

This value is significantly higher than the rate of change calculated for the previous interval from x = 3 to x = 5. We can only conclude that the rate of change or slope of the graph must be increasing and is not constant over an interval $\Delta x = x_2 - x_1$. Look at the graph of the function $f(x) = 3 \cdot x^3$ to understand how this might be so:



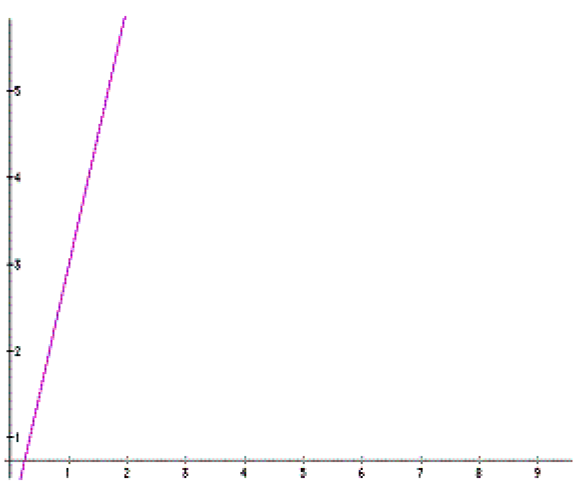
Since the rate of change of a function can change, then we have to come up with a more refined definition of rate of change. We can define the average rate of change of a function over an interval $\Delta x = x_2 - x_1$, to be equal to

$$\textbf{Average rate of change} = \frac{\Delta f}{\Delta x} = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

In the next section we will take a closer look at how we can define the exact or instantaneous rate of change of a function.

Section 4.3 - Instantaneous Rate of Change

In this section we will take a much closer look at rate of change and see how we can define the instantaneous rate of change of a function at any point or value of the independent variable. Let us begin with a study of the simple $f(x) = 3x$. Functions of the form $f(x) = cx$, where c is a constant, express direct relationships. This is because the value of the function, *f*, is a constant multiple or fraction of the independent variable. Thus f is said to be **directly proportional** to x. The graph of the function $f(x) = 3x$ looks like



From x=1 to x=3, Δ x = 2, and Δ f = 9 - 3 = 6. Thus, over this interval of Δ x = 2, Δ f equals 6. The *average rate at which f changed* with respect to x is by definition, $\frac{\Delta f}{\Delta x} = \frac{6}{2} = 3$. For each unit change in x, the change in f is 3. This tells us that f is changing three times faster that x is changing over the interval from x=1 to x=3. We can now look at the interval from x=2 to x=4 where Δ x equals 2.

$$\Delta f = f(x_1 + \Delta x) - f(x_1)$$
$$\Delta f = f(2+2) - f(2)$$
$$\Delta f = 3\cdot(4) - 3\cdot(2)$$
$$\Delta f = 6$$

Thus

$$\frac{\Delta f}{\Delta x} = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \frac{6}{2} = 3$$

Over this interval, the rate of change is the same constant, 3. This leads us to conclude that the rate of change of the function over **any** interval is a constant, 3. This can be proven by the definition of rate of change:

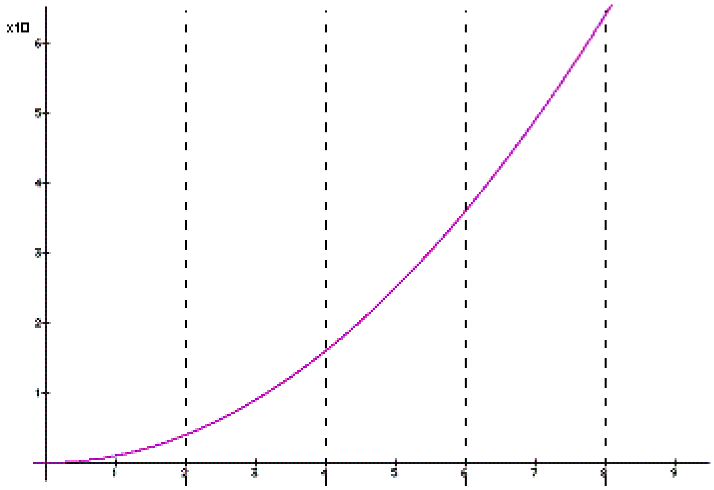
$$f(x) = 3x$$
$$f(x+\Delta x) = 3(x+\Delta x)$$
$$\frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{3\cancel{x} + 3\Delta x - 3\cancel{x}}{\Delta x}$$
$$\frac{\Delta f}{\Delta x} = \frac{3\Delta \cancel{x}}{\Delta \cancel{x}} = 3$$

The rate of change of the direct function $f(x) = cx$ is c and is constant over an interval, Δ x. The graph of a function $f(x) = cx$ is therefore an increasing straight line with a constant slope or steepness equal to **c**. For this reason functions of the form $f(x) = cx$ are also called **linear functions** since their graphs are straight lines with a **constant** rate of change.

On the other hand, the graph of the function $f(x) = x^2$

is not a straight line. Unlike a line, the rate of change of $f(x)$

is not the same constant over any interval Δ x. To define the rate of change for the function we will have to derive a more precise way of defining rate of change of function. To do this we will analyze the function over small intervals of Δ x



The first point to notice is that the rate of change of the function varies with x. When x is small, *f* does not change that much as compared to how much it changes when x is large. We can conclude that the rate of change of *f* with respect to *x*, is not constant over any interval, Δ x, but varies with *x*.

To begin our analysis let us divide up the graph into intervals of $\Delta x = 2$ and study what is happening in each of these intervals separately.

From $x=0$ to $x=2$, $\Delta x=2$, the change in f, Δf, is $f(2) - f(0)= 4$.

From $x=6$ to $x=8$, $\Delta x = 2$, the change in f, Δf, is $f(8) - f(6) = 28$.

Clearly, as x increases the rate of change of the function is increasing and is not a constant as in our study of the line where the function changed at the same rate as x increased.

Returning back to the graph of $f(x) = x^2$; we calculated that from $x=0$ to $x=2$ the change in *f* was 4. We can then conclude that from x=0 to x=2 the **average rate of change** of the function over that particular interval is 2 or:

$$\frac{\Delta f}{\Delta x} = \frac{4}{2} = 2$$

Remember it is called average because this rate of change is only valid from x=0 to x=2.

Now let us consider the interval from x=2 to x=4 where once again the change in x, Δx, is 2. The change in *f* of the graph is equal to:

$$\begin{aligned}\Delta f &= f(x+\Delta x) - f(x)\\ &= f(2+2) - f(2)\\ &= f(4) - f(2)\\ &= 16 - 4\\ &= 12\end{aligned}$$

Thus the average rate of change of the function over this interval is equal to

$$\frac{\Delta f}{\Delta x} = \frac{12}{2} = 6$$

This value is greater than the value we observed from x=0 to x=2. This implies that from x=2 to x=4, f(x) is increasing at a greater rate than from x=0 to x=2. This is despite the fact that in both cases the Δx = 2. The rate of change is therefore not constant over the but is increasing with *x*. When we say the rate of change of *f (x)* from *x=2* to *x=4* is 6, it is only the average value for the given interval since it assumes rate of change is **constant** over that interval.

Let us move on to the next interval of x=4 to x=6. Once again the change in x or Δx equals 2 but the corresponding change in f(x) is not the same as before.

$$\begin{aligned}
\Delta f &= f(x + \Delta x) - f(x) \\
&= (x + \Delta x)^2 - (x)^2 \\
&= \cancel{x^2} + 2x\Delta x + \Delta x^2 - \cancel{x^2} \\
&= 2x\Delta x + \Delta x^2
\end{aligned}$$

For x equal to 4 and *$\Delta x = 2$*, the change in the function over this interval is

$$\Delta f = 2x\Delta x + \Delta x^2$$
$$\Delta f = 2(4)(2) + (2)^2$$
$$\Delta f = 20$$

Note that this corresponds to the same value we would get by:

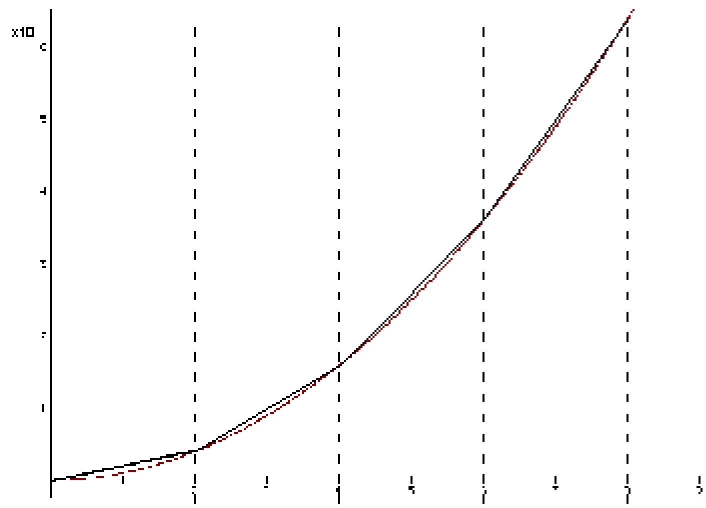
$$\begin{aligned}
\Delta f &= f(6) - f(4) \\
&= 6^2 - 4^2 \\
&= 36 - 16 \\
&= 20
\end{aligned}$$

The average rate of change over this interval is therefore:

$$\frac{\Delta f}{\Delta x} = \frac{20}{2} = 10$$

This is still larger than the rate of change for the previous interval which was 6. Remember rate of change , by definition, refers to how much the function changes with respect to a **change** in the independent variable. The steepness or slope of the line over the interval provides a geometric understanding for this concept.

The average rate of changes calculated over each interval can be used to approximate the graph of *f(x)* from *x=0 to x=8*.



This roughly corresponds to the original graph, but as we can see the rate of change or slope is not constant through out the interval from *x=0 to x=8,* but increases as x-increases. In order to get more accurate answers we need to reduce our interval of $\Delta x=2$ to a much smaller one. The idea being we need to analyze our graph over a small interval, Δx, to see what exactly is going on at each instant the function is changing.

Here is where we begin our study of Calculus. We break down and freeze a changing situation into an infinite series of actions and analyzing what is going on in each individual actions.

Section 4.4 - The Derivative

Calculus involves analyzing instantaneous changes with reference to the entire system. We saw how using a small interval of $\Delta x=2$ caused great differences in the rate of change of the function over the same interval. The function did not **change** nearly as much from x=0 to x=2 as it did from x=4 to x=6. The rate at which the function is changing must be **dependent** on the value of the function. In other words the rate of change must be defined at each interval and this value must be unique to that interval.

We can approximate the graph from *x=6 to x=8* more closely by using a smaller interval of $\Delta x=0.5$. This will give an even more accurate description of the graph of the function $f(x) = x^2$ over that interval. Keep in mind that the rate of change of the function varies with x, and the only way to accurately analyze this change is by breaking the graph up into smaller and smaller sub-intervals and study what is happening over these infinitely small intervals:

First let us define a method for calculating the rate of change between any two points on the graph of *f(x):* If the change in *f* of a function from a point *x* to another point $(x+ \Delta x)$ is given by:

$$\Delta f = f(x+\Delta x) - f(x)$$

The rate of change of *f(x)* over this interval is found by dividing by Δx.

$$\frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Remember the Δx on the left side is the same as the Δx in the $f(x + \Delta x)$ expression.

From *x=6 to x=6.5* the rate of change is:

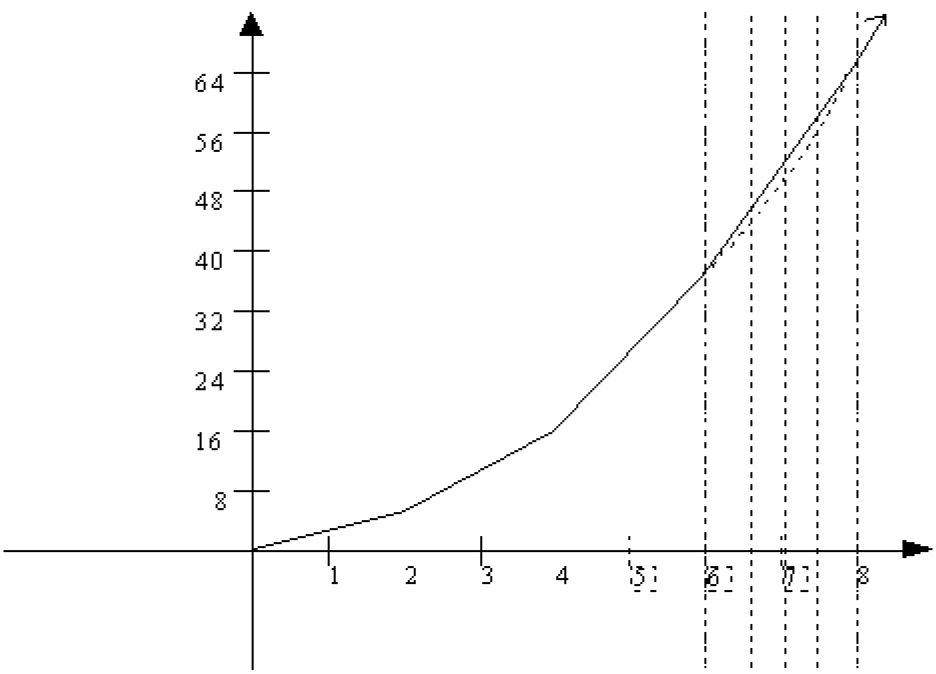
$$\frac{\Delta f}{\Delta x} = \frac{f(6+0.5) - f(6)}{0.5} = 12.5$$

From x=6.5 to x=7 the rate of change is:

$$\frac{\Delta f}{\Delta x} = \frac{f(6.5+0.5) - f(6.5)}{0.5} = 13.5$$

Similarly the rate of change *from x=7 to x=7.5 and x=7.5 to x=8* is **14.5** and **15.5** respectively. Notice how the rate of change of the function *slightly* increases as x increases.

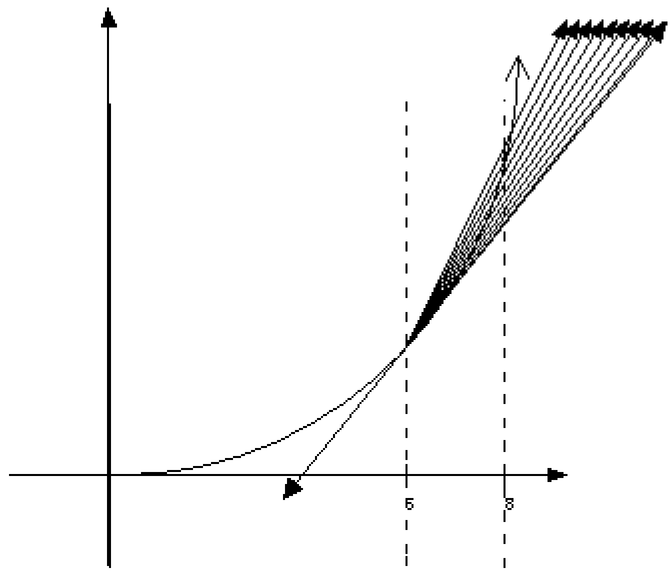
This implies that there must be a way to **instantaneously** define the rate of change of function through "*one point*" on the graph. Before continuing, take a look at how the average rate of changes or slopes just calculated, approximate the graph from *x=6 to x=8.*



As you can see from the above graph, the smaller interval of *Δx=0.5* gives a more accurate approximation of the graph of f(x). Our results are only averages because the rate of change of *f(x)* is increasing with *x* and is not constant over an interval *Δx*. To define the **instantaneous** rate of change of we need to define *Δx* as an infinitely small distance separating two point *x* and *(x+ Δx)* on the graph of *f(x).* The rate of change calculated through these two points will then give us the exact rate of change of *f(x)* at that point, *x*. Since through any one point there exists an infinite many lines, we are in actuality giving the slope of the line through the two points *x* and *(x+Δx)* where *Δx* goes to zero. The two points are assumed to be infinitely close to each other such that the rate of change *f(x)* over that interval is constant.

As we let these two points *(x)* and *(x+ Δx)* come closer and closer together, the slope of the line through these two points corresponds to the actual graph itself. For example with Δx=2, the line through x = 6 an d x= 8 only roughly represented the actual graph. However by breaking up the interval from x=6 to x=8 into four sub-intervals of *Δx= 1/2*, we got a better approximation to the actual graph itself.

The instantaneous rate of change can now be defined as the rate of change of the line through a point x and *(x+ Δx)* where *Δx* is near to zero. This is called taking the limit as Δx goes to zero. In the graph this looks like.



By letting the difference or Δx between the two points go to zero we are able to accurately describe the behavior of the function at any *point* on the graph. With Δx large we can only approximate the functions behavior over the interval. This infinitely small Δx allows us to define the rate of change of f(x) through any one point on the graph, where point is by definition an infinitely small interval between x and x+ Δx over which the rate of change is assumed to be constant or does not change.

Thus the instantaneous rate of change can also be thought of as the tangent to the graph at any point, where point is actually two simultaneous points on the graph, separated by an infinitely small distance.

For example the tangent to the graph at x=6 is given by:

$$\frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{(6+\Delta x)^2 - (6)^2}{\Delta x}$$

$$\frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cancel{36} + 12\Delta x + \Delta x^2 - \cancel{36}}{\Delta x}$$

$$\frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} 12 + \Delta x$$

$$\frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} 12 + 0 = 12$$

We can now go on to define the rate of change of the at any point x of the function *f(x) = x^2* as:

$$\frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
$$\frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} 2x + \Delta x$$
$$\frac{\Delta f}{\Delta x} = 2x$$

This function is called the **derivative** and gives us the rate of change or tangent to graph of

$f(x) = x^n$ at any point x. It is extremely important to realize that the function's derivative is also a function of x. In other words its value changes with x to reflect a change in the rate of change of the original function. This may sound a bit confusing but I encourage you to spend a few minutes and think how it is so.

You are probably asking yourself how can we assume that the derivative is constant over an infinitely small interval Δx ? Since the function's rate of change is changing then it can only be constant over an infinitely small interval or point. It is through this point that we define the derivative by taking the limit as Δx goes to zero. In effect we define the graph of the function through a series of connected lines with changing slopes. In the following paragraph we will mathematically prove that the derivatives value converges to the exact value as all errors go to zero as Δx goes to zero. Since the graph of the function is continuous over an infinitely small interval then so will its derivative be defined for that interval or point.

Note that Δx and Δf no longer refer to discrete values. When we let Δx go to zero, its value becomes infinitely small such that:

$$\frac{df}{dx} = \lim_{\Delta x \to o} \frac{\Delta f}{\Delta x}$$

Here ***df*** and ***dx*** represent an infinitely small difference in x, where:

$$dx \to 0$$

We'll now see how to find the derivative of any simple function, $f(x) = x^n$. Since the derivative is also a function , we shall call it *f'(x),* "f prime of x" whe $f'(x) = \frac{df}{dx}$

Our results can now be generalized for any function $f(x) = x^n$ where n is a positive integer.

$$f'(x) = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
$$f'(x) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - (x)^n}{\Delta x}$$

From the Binomial expansion theorem we get: (The theorem is just algebra and you can find a proof of it by clicking here).

$$f'(x) = \lim_{\Delta x \to 0} \frac{\cancel{x^n} + nx^{n-1}\Delta x + \frac{(n)(n-1)}{2!}x^{n-2}\Delta x^2 + \ldots\ldots \frac{(n)(n-1)(n-2)\ldots(n-(n-1))}{n!}x^0\Delta x^n - \cancel{x^n}}{\Delta x}$$

This simplifies to

$$f'(x) = \lim_{\Delta x \to 0} \frac{nx^{n-1}\Delta x + \frac{(n)(n-1)}{2} x^{n-2}\Delta x^2 + \frac{(n)(n-1)(n-2)x^{n-3}\Delta x^{3\ n}}{3!}}{\Delta x}$$

We can replace all the $\frac{(n)(n-1)}{2!}$ $\frac{(n)(n-1)(n-2)}{3!}$ $c_1, c_2 \ldots c_n$ $c_1, c_2 \ldots c_n$ are constants:

$$f'(x) = \lim_{\Delta x \to 0} \frac{nx^{n-1}\Delta x + c_1 x^{n-2}\Delta x^3 \ldots + c_n x^0 \Delta^n}{\Delta x}$$

Take note of how x goes from x to n'th power to the 0 power while Δx goes from the 0 power to the n'th power in the last term. If we divide through by Δx we get:

$$f'(x) = \lim_{\Delta x \to 0} nx^{n-1} + c_1 x^{n-2}\Delta x + c_2 x^{n-3}\Delta x \ldots\ldots c_n x^0 \Delta x^n$$

As we take the limit as Δx goes to zero, every term after the first one goes to zero and hence cancels out; This leaves us with:

$$f'(x) = nx^{n-1} + 0 + +0 + 0\ldots$$

$$f'(x) = \frac{df}{dx} = n \cdot x^{n-1}$$

The rate of change of a function $f(x) = x^n$ at any point x is given by the function's derivative $f'(x)$ evaluated at that point x.